Hamiltonian properties of 3-connected {claw,hourglass}-free graphs

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Abstract

We consider sufficient conditions for hamiltonian properties, namely, the well-known forbidden subgraph conditions for hamiltonicity and Hamilton-connectedness and degree conditions for circumference and hamiltonicity in 3-connected claw-free graphs, and we show that these conditions can be substantially strengthened under an additional assumption that the graph under consideration is hourglass-free (where hourglass is the graph with degree sequence $4, 2, 2, 2, 2$).

1 Introduction

By a *graph* we always mean a simple finite graph $G = (V(G), E(G))$. We use $\delta(G)$ to denote the minimum degree of G, and, for a positive integer k, we set $\sigma_k(G) = \min\{\sum_{x\in I} d_G(x) |\ I \subset V(G)$ independent, $|I| = k\}$ if G contains an independent set of size k, and $\sigma_k(G) = \infty$ otherwise. If G is the line graph of a graph H, we denote $G = L(H)$ and $H = L^{-1}(G)$.

Throughout the paper, $c(G)$ denotes the *circumference of* G, i.e., the length of a longest cycle in G. A graph G is *hamiltonian* if $c(G) = |V(G)|$, i.e., if G contains a *hamiltonian cycle*, *Hamilton-connected* if, for any $x, y \in V(G)$, G contains a *hamiltonian* (x, y)*-path*, i.e., an (x, y)-path containing all vertices of G, and 1-*Hamilton-connected* if $G - x$ is Hamilton-connected for any $x \in V(G)$.

If $\mathcal{F}$ is a family of graphs, we say that G is $\mathcal{F}$*-free* if G does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$, and the graphs in $\mathcal{F}$ are called *forbidden induced subgraphs* in this context. Specifically, for $\mathcal{F} = \{K_{1,3}\}$, we say that G is *claw-free*. Throughout, P_i denotes the path on i vertices. Graphs often used as forbidden induced subgraphs are shown in Fig. 1; here the graph Γ_0 is called the *hourglass*, $B_{i,j}$ the *generalized bull* and $N_{i,j,k}$ the *generalized net* $(i, j, k \geq 1)$.

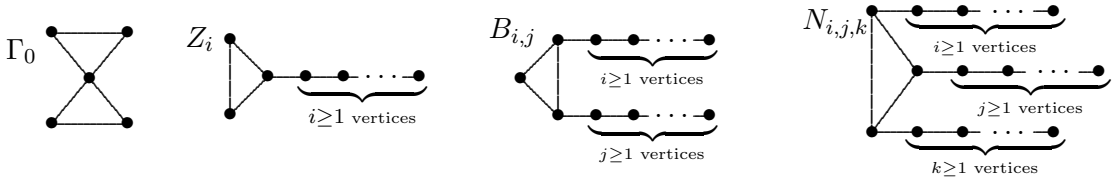


Figure 1: The graphs Γ_0, Z_i, $B_{i,j}$ and $N_{i,j,k}$

For a set $X \subset E(G)$, an X*-contraction of* G is the graph $G|_X$ obtained from G by identifying the vertices of each edge in X and removing the resulting loops. If F and G are graphs, we say that G *has a nontrivial contraction to* F if there is $X \subset E(G)$ such that $G|_X \simeq F$ and for every $v \in V(F)$, at least two vertices of G are contracted to v.

For $x \in V(G)$, the *local completion of* G *at* x is the graph $G_x^* = (V(G), E(G) \cup \{uv |\ u, v \in N_G(x)\})$ (i.e., in G_x^*, $N_{G_x^*}(x)$ induces a complete graph). The *closure* of a claw-free graph G is the graph $\mathrm{cl}(G)$ obtained from G by recursively performing the local completion operation at vertices with connected noncomplete neighborhood, as long as this is possible (more precisely, there is a sequence of graphs $G_1, \ldots, G_k$ such that $G_1 = G$, $G_{i+1} = (G_i)_x^*$ for some vertex $x \in V(G_i)$ with connected noncomplete neighborhood, $i = 1, \ldots, k-1$, and $G_k = \mathrm{cl}(G)$). The following result summarizes basic properties of the closure operation.

Theorem A [10]. *Let G be a claw-free graph. Then*
- *(i)* $\mathrm{cl}(G)$ *is uniquely determined,*
- *(ii)* $c(\mathrm{cl}(G)) = c(G)$,
- *(iii)* $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

There are many results on hamiltonian properties of graphs in classes defined in terms of forbidden induced subgraphs. In this paper, we will consider these questions in 3-connected graphs. We first summarize some known results.

Theorem B. *Let G be a 3-connected claw-free graph.*
- *(i)* **[9]** *If G is P_{11}-free, then G is hamiltonian.*
- *(ii)* **[6]** *If G is Z_8-free, then G is hamiltonian.*
- *(iii)* **[3]** *If G is Z_9-free, then either G is hamiltonian, or G is isomorphic to the line graph of the graph obtained from the Petersen graph by adding one pendant edge to each vertex.*
- *(iv)* **[12, 4]** *If G is $N_{i,j,k}$-free with $i + j + k \leq 9$, then G is hamiltonian.*

Theorem C [1]. *Let G be a 3-connected $\{K_{1,3}, P_9\}$-free graph. Then G is Hamilton-connected.*

We also list here the best known degree conditions for circumference and hamiltonicity.

Theorem D [8]. *Let G be a 3-connected claw-free graph. Then $c(G) \geq \min\{6\ \delta(G) - 15, n\}$.*

Theorem E [7]. *Let G be a 3-connected claw-free graph of order $n \geq 363$ such that $\delta(G) \geq \frac{n+34}{12}$. Then either G is hamiltonian or $G \in \mathcal{F}_1 \cup \mathcal{F}_2$.*

For a detailed description of the classes $\mathcal{F}_1$ and $\mathcal{F}_2$ of Theorem E we refer to [7]; here we only note that for every graph $G \in \mathcal{F}_1 \cup \mathcal{F}_2$, $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

There are results indicating that some conditions for hamiltonian properties can be improved under an additional assumption that the graph under consideration is hourglass-free. For example, it is a well-known fact, observed independently by several authors (see e.g. [2]), that the Matthews-Sumner conjecture (every 4-connected claw-free graph is hamiltonian) is true in Γ_0-free graphs (and even for only 4-edge-connected graphs [11]); moreover, it was shown recently [5] that every 4-connected $\{K_{1,3}, \Gamma_0\}$-free graph is 1-Hamilton-connected, and 1-Hamilton-connectedness is polynomial in the class of $\{K_{1,3}, \Gamma_0\}$-free graphs.

In this paper, we show that Theorems B, C, D and E can be substantially strengthened under an additional assumption that G is Γ_0-free.

2 Results

Our first result strengthens Theorem B in the case of Γ_0-free graphs.

Theorem 1. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph. If G is*
- *(i)* *P_{20}-free, or*
- *(ii)* *Z_{18}-free, or*
- *(iii)* *$N_{2i,2j,2k}$-free with $i + j + k \leq 9$,*

then G is hamiltonian.

We further have the following strengthening of Theorem C.

Theorem 2. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0, P_{12}\}$-free graph. Then G is Hamilton-connected.*

Our next result shows that the minimum degree bound on circumference given in Theorem D can be also substantially strengthened in the case of Γ_0-free graphs.

Theorem 3. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n. Then either*

$$c(G) \geq \min\{\sigma_{12}(G), n\},$$

or $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

Theorem 3 immediately implies the following corollary.

Corollary 4. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n. Then either*

$$c(G) \geq \min\{12\ \delta(G), n\},$$

or $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

The following corollary gives a weaker bound, but without any exception class.

Corollary 5. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n. Then*

$$c(G) \geq \min\{\sigma_9(G), n\}.$$

Finally, we have the following strengthening of Theorem E.

Theorem 6. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n such that*

$$\sigma_{13}(G) \geq n + 1.$$

Then either G is hamiltonian, or $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

Theorem 6 has the following immediate consequence.

Corollary 7. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n such that $\delta(G) \geq \frac{n+1}{13}$. Then either G is hamiltonian, or $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.*

The following consequence of Theorem 6 has stronger assumptions, but no exception class.

Corollary 8. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n such that $\sigma_{10}(G) \geq n+1$. Then G is hamiltonian.*

3 Sharpness

Let H be the subdivision of the Petersen graph (see Fig. 2(a)), and let $G = L(H)$. Then G is a 3-connected $\{K_{1,3}, \Gamma_0\}$-free nonhamiltonian graph which is P_{21}-free, Z_{19}-free and $N_{i,j,k}$-free for $i + j + k \geq 19$. This example shows that parts (i), (ii) and (iii) of Theorem 1 are sharp. Note that the sharpness example can be extended to an infinite family by adding arbitrary number of pendant edges to vertices of degree 3 in H.

We think that Theorem 2 is not sharp, and we believe that the following is true.

Conjecture 9. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0, P_{16}\}$-free graph. Then G is Hamilton-connected.*

Let H be the subdivision of the Wagner graph (see Fig. 2(b)), and let $G = L(H)$. Then it is easy to verify that G is 3-connected and $\{K_{1,3}, \Gamma_0, P_{17}\}$-free, but not Hamilton-connected (the graph G has no hamiltonian path joining the vertices that correspond to the edges $e, f \in E(H)$

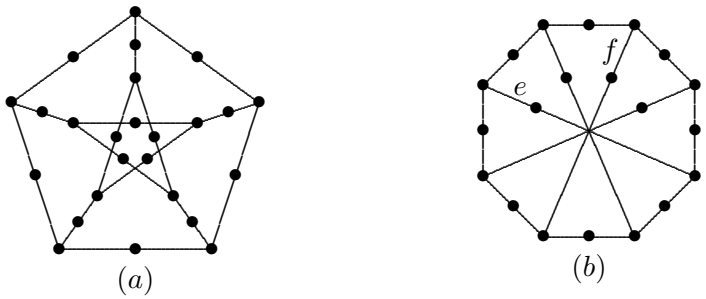


Figure 2: Subdivisions of the Petersen graph (a) and of the Wagner graph (b)

indicated in Fig. 2(b)). This example shows that Conjecture 9, if true, is sharp. Note that this example can be also extended to an infinite family by adding pendant edges to vertices of degree 3.

Let again H be the subdivision of the Petersen graph, and let $G = L(H)$. It is easy to check that $c(G) = 27$, while $n = 30$. Since also $\sigma_9(G) = 27$, we have $c(G) = \sigma_9(G) < n$, which shows that Corollary 5 is sharp. Similarly, $\sigma_{10}(G) = 30 = n$ and G is nonhamiltonian, which shows that also Corollary 8 is sharp. Note that, in this case, adding pendant edges does not create an infinite family of sharpness examples, and we admit that Corollaries 5 and 8 could be possibly improved under an additional assumption that G is sufficiently large.

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