Incidence coloring of some special classes of graphs

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Abstract

Incidence coloring was defined by Brualdi and Massey [2] as a tool to study strong edge colorings of bipartite graphs. However, soon after its definition, the coloring itself attracted the attention of several researchers from different points of view.

An *incidence* in a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident to v. Two incidences (v, e) and (u, f) are *adjacent* if at least one of the following holds: (i) $v = u$, (ii) $e = f$, or (iii) edge vu is from the set $\{e, f\}$. An *incidence coloring* of G is a coloring of its incidences assigning distinct colors to adjacent incidences. The corresponding chromatic number is called the *incidence chromatic number* of G, denoted by $\chi_i(G)$.

Brualdi and Massey [2] conjectured that $\chi_i(G) \leq \Delta(G) + 2$ for any graph G, where $\Delta(G)$ denotes the maximum degree of G. The conjecture was disproved by Guiduli [3], who showed that Paley graphs with maximum degree Δ have incidence chromatic number at least $\Delta + \Omega(\log \Delta)$. However, for many of the commonly considered graph classes the incidence chromatic number is bounded by $\Delta + c$ for some constant c, and several papers are devoted to the proof of this type of result, including the following one.

Theorem 1 ((Maydanskiy, 2005))**.** *Five colors suffice for an incidence coloring of any subcubic graph.*

In order to obtain upper bounds on the incidence chromatic number, in many cases, stronger statements concerning incidence colorings with further local constraints are proved, allowing to apply induction in a more efficient way.

An incidence coloring of a graph G using k colors is an *incidence* (k, p)*-coloring* of G if for every vertex v of G, the number of colors used for coloring the incidences of the form (u, uv) is at most p.

Hosseini Dolama, Sopena and Zhu [5] proved that every planar graph with maximum degree Δ admits an incidence $(\Delta + 7, 7)$-coloring and, thus, has incidence chromatic number at most $\Delta + 7$. This bound was further improved to $\Delta + 4$ for triangle-free planar graphs [6], to $\Delta + 3$ (resp. $\Delta + 2$, $\Delta + 1$) for planar graphs of girth at least 6 (resp. 11, 16) [6]. The last result was further improved to girth 14 [1].

Some of these results were proved for more general graph classes, namely graphs with bounded maximum average degree. The *average degree* of a graph G is the mean value of the degrees of its vertices. The *maximum average degree* $\mathrm{mad}(G)$ of a graph G is then defined as the maximum value of the average degrees of its subgraphs. When G is a planar graph with girth g, it is folklore to establish the inequality $\mathrm{mad}(G) < \frac{2g}{g-2}$.

In [6] the authors proved the following result.

Theorem 2 ((Hosseini Dolama, Sopena, 2005))**.** *Let G be a graph with $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 5$. Then G admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.*

We extended this result to $\mathrm{mad}(G) < 3$ and $\Delta(G) = 4$:

Theorem 3. *Let G be a graph with $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 4$. Then G admits a $(\Delta(G)+2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.*

Moreover, we proved another result for graphs with larger maximum average degree:

Theorem 4. *Let G be a graph with* $\mathrm{mad}(G) < \frac{10}{3}$ *and* $\Delta(G) \geq 8$. *Then G admits a* $(\Delta(G)+2, 2)$*-incidence coloring. Therefore,* $\chi_i(G) \leq \Delta(G) + 2$.

Recall that the *star arboricity* of an undirected graph G is the smallest number of star forests needed to cover G. Yang [8] observed the following: let G be an undirected graph with star arboricity $st(G)$, let $s : E(G) \to \{1, \ldots, st(G)\}$ be a mapping such that $s^{-1}(i)$ is a forest of stars for every i, $1 \leq i \leq st(G)$, and let λ be a proper edge coloring of G. Now define the mapping f by $f(u, uv) = s(uv)$ if v is the center of a star in some forest $s^{-1}(i)$ (if some star is reduced to one edge, we arbitrarily choose one of its end vertices as the center) and $f(u, uv) = \lambda(uv)$ otherwise. It is not difficult to check that f is indeed an incidence coloring of G. Therefore, thanks to the classical result of Vizing, the relation $\chi_i(G) \leq \Delta(G) + st(G)$ (resp. $\chi_i(G) \leq \Delta(G) + st(G) + 1$) holds for every graph of class 1 (resp. of class 2). The facts that planar graphs with $\Delta \geq 7$ are class 1 [7] and that the star arboricity of any planar graph is at most 5 [4] led to the following result.

Theorem 5 ((Yang, 2007))**.** *If G is a planar graph with* $\Delta(G) \neq 6$, *then* $\chi_i(G) \leq \Delta(G) + 5$. *If* $\Delta(G) = 6$, *then* $\chi_i(G) \leq \Delta(G) + 6$.

Yang [8] proposed the following question: Are $\Delta(G) + 5$ colors enough for graphs with maximum degree 6? We state the following:

Theorem 6. *If G is a graph with* $\Delta(G) \leq 6$ *and with no* 6*-regular component on an odd number of edges, then* $\chi_i(G) \leq \Delta(G) + 4$.

As a consequence of the previous theorem, we positively answer Yang's question about planar graphs with maximum degree 6, even improving the suggested bound:

Corollary 1. *Every planar graph G with* $\Delta(G) = 6$ *satisfies* $\chi_i(G) \leq \Delta(G) + 4$.

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