Online coloring of intersection graphs of disks, polygons and other convex shapes

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Abstract

We introduce an online coloring algorithm for intersection graphs of various geometrical shapes such as disks, parallelograms, triangles, general convex shapes. We consider intersection graphs of congruent geometrical shapes and we assume the ratio between largest and smallest shape is bounded (say by a real number σ). The competitive ratio of our algorithms is $O(\log \sigma)$. Our algorithm uses a Hadwiger-Nelson type coloring of the plane specific to each shape.

Intersection graphs of families of geometric objects attracted much attention both for their theoretical properties and practical applications (c.f. McKee and McMorris [8]). For example intersection graphs of families of disks, and in particular disks of unit diameter (called *unit disk intersection graphs*), play a crucial role in modeling radio networks. We say that a graph coloring algorithm is *online* if the input graph is not known a priori, but is given vertex by vertex (along with all edges adjacent to already revealed vertices). Each vertex is colored at the moment when it is presented and its color cannot be changed later. On the other hand, *offline* coloring algorithms know the whole graph before they start assigning colors. The online coloring can be much harder than offline coloring, even for paths. For an online coloring algorithm by the *competitive ratio* we mean the worst-case ratio of the number of colors used by this algorithm to the chromatic number of the graph. For unit disk intersection graphs First-Fit algorithm has competitive ratio 5 [1, 9, 10].

Fiala, Fishkin and Fomin [3] presented a polynomial-time online algorithm that finds an $L(2,1)$-labeling of an intersection graph of disks of bounded diameter. $L(2,1)$-labeling is an assignment of integers to the vertices in such a way that labels of adjacent vertices differ by at least 2, and vertices at distance two get different labels. The algorithm is based on a special coloring of the plane, that resembles colorings studied by Exoo [2], inspired by the classical Hadwiger-Nelson problem [5]. A similar idea of reserving a set of colors for upcoming vertices can be found in a paper by Kierstead and Trotter [7]. The paper [6] presents an algorithm for coloring unit disk intersection graph. It is inspired by [3], however, a b-fold coloring of the plane (see [4]) is used instead of a classical coloring, what allowed obtaining competitive ratio smaller than 5.

In this paper we generalize results from [6] for intersection graphs of disks with bounded diameter and other geometrical convex shapes. The competitive ratio of our algorithms is $O(\log \sigma)$, improving $O(\sigma^2)$ from [3]. Throughout the paper we always assume that the input intersection graph is given along with its geometric representation.

We present our algorithm for intersection graphs of disks with a bounded diameter, namely with a diameter in the interval $[\rho, \sigma]$. Our algorithm uses a special coloring of the plane. By $G_{[\rho,\sigma]}$ we denote an infinite graph with $\mathbb{R}^2$ as the vertex set and edges between pairs of points at euclidean distance within $[\rho, \sigma]$. A function $\varphi : \mathbb{R}^2 \longrightarrow [k]$ is called a coloring of $G_{[\rho,\sigma]}$ with colors $[k]$ if for any two points $p_1, p_2 \in \mathbb{R}^2$ at euclidean distance within $[\rho, \sigma]$ holds $\varphi(p_1) \neq \varphi(p_2)$. A coloring φ

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of $G_{[\rho,\sigma]}$ is called *solid* if there exists a tiling such that each tile is monochromatic and tiles in the same color are at distance greater than σ. An example of a solid 1-fold 7-coloring of the plane (by John Isbell [11]) is presented in Figure 1.

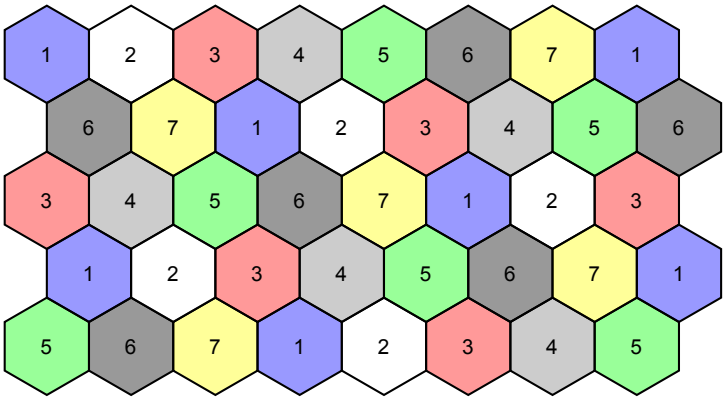


Figure 1: 7-coloring of the plane for $\sigma = \sqrt{7}/2 \approx 1.32$

Now we are ready to introduce the online algorithm for coloring disk intersection graphs presented by Fiala, Fishkin and Fomin [3]. The input of the algorithm is a solid coloring of $G_{[1,\sigma]}$ and a sequence of the disks of diameters within $[1, \sigma]$. The algorithm is online so it colors the vertex v_i corresponding to the disk D_i knowing only disks $D_1, \ldots, D_i$ and the color cannot be changed later on. The idea of the algorithm is the following: find the tile containing the center of D_i. The color of a vertex corresponding to D_i is the smallest available color equal modulo k to the color of the tile containing the center of D_i.

Algorithm 1: $Color_\varphi((D_i)_{i\in[n]})$

foreach $i \in [n]$ **do**
  Read D_i, let v_i be the center of D_i
  let $T(v_i)$ be the tile containing v_i
  $t(v_i) \leftarrow |\{v_1, \ldots v_{i-1}\} \cap T(v_i)|$
  $c(v_i) \leftarrow \varphi(v_i) + k \cdot t(v_i)$
return c

The algorithm $Color_\varphi$ uses at most $k\omega(G)$ colors, where k is the number of colors used by coloring φ so its competitive ratio is at most k [3]. Desired colorings of the plane are given in papers [3, 4]. The numbers of used colors by these colorings are proportional to the square of the parameter σ, so competitive ratio of the above algorithm $Color_\varphi$ is $O(\sigma^2)$.

Now we present how to reduce the competitive ratio to $O(\log \sigma)$. For $p \in \mathbb{N}_+$ we divide the interval $[1, \sigma]$ into p intervals: $[1, \sigma^{1/p}), [\sigma^{1/p}, \sigma^{2/p}), [\sigma^{2/p}, \sigma^{3/p}), \ldots, , [\sigma^{(p-1)/p}, \sigma]$. Notice that for disks with diameter in $[\sigma^{i/p}, \sigma^{(i+1)/p}]$ the ratio between their diameters does not exceed $\sigma^{1/p}$. Our modification of the algorithm is as follows: take a solid coloring φ of $G_{[1,\sigma^{1/p}]}$. Such coloring is given in [4]. By scaling φ we can obtain a coloring φ_i of $G_{[\sigma^{i/p},\sigma^{(i+1)/p}]}$ with the same number of colors say k. We reserve different set of colors for φ_i for every $\in \{1, \ldots, p\}$ (so we reserved $p \cdot k$ colors). When the new disk D is revealed, say of diameter d, we find $i \in \{1, \ldots, p\}$ such that $d \in [\sigma^{i/p}, \sigma^{(i+1)/p})$.

We color D with the algorithm $Color_{\varphi_i}$ (based on the i-coloring φ_i). The modified algorithm uses at most $p \cdot k \cdot \omega(G)$ colors where k is the number of colors used by the coloring φ. There exists a constant C such that $k = C \cdot \sigma^{2/p}$. We choose p to minimize $p \cdot k = p \cdot C \cdot \sigma^{2/p}$ (see [3, 4]). Taking $p = \lceil \log \sigma \rceil$ the number of colors used by modified algorithm is $O((\log \sigma) \cdot \omega(G))$ and as the result we get:

Theorem 1. *There exists an online algorithm coloring intersection graphs of disks with diameters in the interval* $[1, \sigma]$ *with competitive ratio* $O(\log \sigma)$.

The modified algorithm can be adopted to intersection graphs of different geometrical shapes. It is enough to use adequate coloring of the plane. For intersection graph of homothetic copies of parallelograms the coloring on the figure 2 can be used, and for intersection graphs of homothetic copies of triangles the coloring on the figure 3. We say that a shape T' is a homothetic copy of a shape T if it can be obtained from T by scaling and traversing (but without rotation).

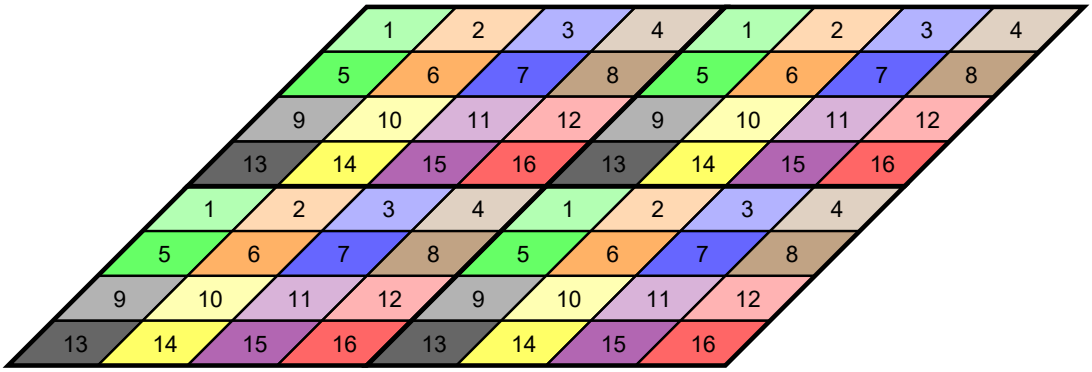


Figure 2: 16-coloring of the plane for parallelogram

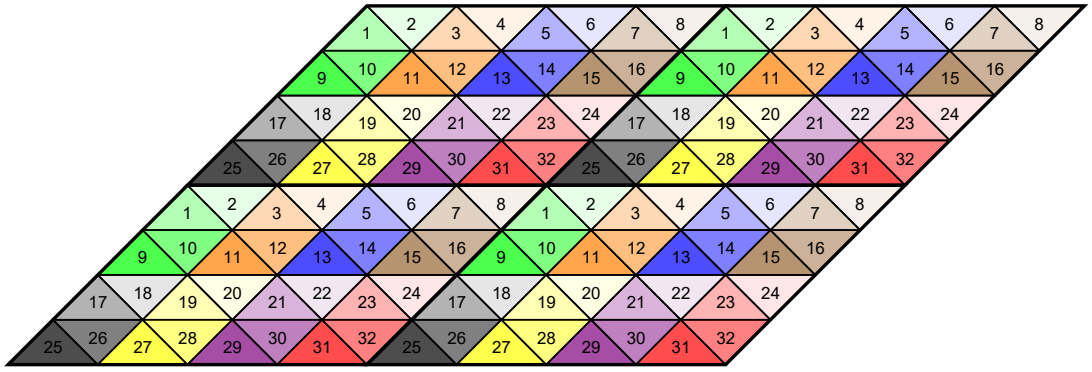


Figure 3: 32-coloring of the plane for triangles

Now consider any convex shape T and an intersection graph of homothetic copies of T. Let P be the smallest parallelogram containing T. Let A_1, A_2, A_3, A_4 be points on each side of P belonging

also to T. If there are more than one point common of T and a side of P then choose among such points the one closes to the middle of the side of P. Let T^d a be a convex hull of the points A_1, A_2, A_3, A_4. Let $C(T)$ be an intersection of segments A_1, A_3 and A_2, A_4. It is easy to prove that T^d is a parallelogram and $T^d \subseteq T$. We can define a solid coloring φ of the plane (analogous to the one on the figure 2) such that for any two homothetic copies T_1, T_2 of the shape T (with ratio bounded by sigma) if $C(T_1)$ and $C(T_2)$ lie in the same tile then T_1, T_2 intersect and if $C(T_1)$ and $C(T_2)$ lie in different tiles of the same color then T_1 and T_2 are disjoint. For such φ we obtain online coloring algorithm for interesection graph of homothetic copies of the shape T, assuming that the ration between diameters of any two copies is bounded by σ, with competitive ration $O(\log \sigma)$.

We may also consider an intersection graph of similar copies of a convex shape T (scaling, transformation and rotation are allowed). To adapt our method we consider two disks D_1 and D_2 with common center.We choose radii of D_1 and D_2 in such a way that when we scale T to the extremes of the given graph D_1 is contained in T and D_2 contains T (we do not scale D_1 and D_2). Now we find a coloring of a plane such that each tile has diameter equal to the diameter of D_1 and tiles of the same color are at distance greater than the diameter of D_2. The number of colors required for such coloring strongly relies on the shape of T and on the ratio of scaling. Nevertheless for a fixed T and the ratio between diameters of any two copies bounded by σ, the coloring algorithm has competitive ratio $O(\log \sigma)$.

Our approach can also be applied for different models of coloring e.g. $L(2,1)$-labeling. For $L(2,1)$-labeling of σ-disks or other convex shapes the competitive ratio of $O(\log \sigma)$ is yet the best, up to our knowledge.

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