Domination problems in grid graphs

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Abstract

We investigate the 2-domination number for grid graphs, that is the size of a smallest set D of vertices of the grid such that each vertex of the grid belongs to D or has at least two neighbours in D. We find the 2-domination number of any $n \times m$ grid. The proof relies on some dynamic programming algorithms, using transfer matrices in (min,+)-algebra. This confirms the results found by Lu and Xu, and Shhaeen et al. for $n \leq 4$ and correct the value of Shaheen et al. for $n = 5$. We apply the same method to solve the Roman domination problem on grids.

1 Introduction

A *dominating set* D in a graph G is a subset of the vertices such that every vertex in $V(G) \setminus D$ is neighbour with at least one vertex in D. The *domination number* of G is the minimum size of a dominating set of G. The grid graph $G_{n,m}$ is the graph with vertex set the cells of a grid with n lines and m columns, and two vertices are adjacent if the corresponding cells are adjacent with the von Neumann neighbourhood. An open problem which remained unsolved for a long time was to find out the domination number of grid graphs. Hare *et al.* [5] and then Fisher [3] proposed dynamic algorithms to compute the domination number of grids of small height. In 1992, Chang [1, 2] made a conjecture for the domination number in grid graphs of arbitrary size by showing an upper bound for this problem: $\gamma(G_{n\times m}) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4$. Finally Gonçalves et al. [4] proved Chang's conjecture in 2011.

We show that the method used in [4] can be used to solve other domination-type problems. We adapt here the method to the 2-domination problem and the Roman domination, and we belive that this method can be used on numerous other parameters on grid-like graphs. A *2-dominating* set of G is a subset $D \subset V(G)$ such that every vertex not in D has at least two neighbours in D. The *2-domination number* of a graph G, denoted $\gamma_2(G)$ is the minimum size of a 2-dominating set of G. Let $\gamma_2(n,m) = \gamma_2(G_{n\times m})$ be the 2-domination number of an $n \times m$ grid.

Theorem 1. *For all* $1 \leq n \leq m$,

$$\gamma_2(n,m) = \begin{cases} \lceil \frac{m+1}{2} \rceil & \text{if } n = 1 \\ m & \text{if } n = 2 \\ m + \lceil \frac{m}{3} \rceil & \text{if } n = 3 \\ 2m - \lfloor \frac{m}{4} \rfloor & \text{if } n = 4 \text{ and } m \mod 4 = 3 \\ 2m - \lfloor \frac{m}{4} \rfloor & \text{if } n = 4 \text{ and } m \mod 4 \neq 3 \\ 2m + \lceil \frac{m}{7} \rceil + 1 & \text{if } n = 5 \text{ and } m \mod 7 \in \{0,6\} \\ 2m + \lceil \frac{m}{7} \rceil & \text{if } n = 5 \text{ and } m \mod 7 \notin \{0,6\} \\ 2m + \lfloor \frac{6m}{11} \rfloor + 1 & \text{if } n = 6 \text{ and } m \mod 11 \in \{0,2,6\} \\ 2m + \lfloor \frac{6m}{11} \rfloor + 2 & \text{if } n = 6 \text{ and } m \mod 11 \notin \{0,2,6\} \\ 3m - \lfloor \frac{m}{18} \rfloor + 1 & \text{if } n = 7 \text{ and } m > 9 \text{ and } m \mod 18 \leq 9 \\ 3m - \lfloor \frac{m}{18} \rfloor & \text{if } n = 7 \text{ and } (m \leq 9 \text{ or } m \mod 18 > 9) \\ 3m + \lceil \frac{m}{3} \rceil & \text{if } n = 8 \text{ and } m \mod 3 = 1 \\ 3m + \lceil \frac{m}{3} \rceil + 1 & \text{if } n = 8 \text{ and } m \mod 3 \neq 1 \\ \lfloor \frac{(n+2)(m+2)}{3} \rfloor - 6 & \text{if } n \geq 9 \end{cases}$$

2 Method for computing $\gamma_2(n, m)$ when $n \leq 13$

In this section we compute the values of $\gamma_2(G_{n,m})$ by a dynamic algorithm for a fixed n. All matrix operations are done in the (min,+)-algebra. If V is a vector of size n is indexed from 0 to $n-1$.In all what follows, the grids we consider always have n lines and m columns.

We refer to a cell in D as containing a *stone*. We first define the notion of *state* for a cell of the grid: in case a cell does not contain a stone, we store some information about the number of neighbours containing a stone the cell has "so far". We set **state 0** for a cell containing a stone, **state 1** for a cell which is adjacent to at least 2 stones, and **state 2** for a cell which has only one neighbour with a stone. The state of a cell depends on its neighbours for which we have decided whether they belong to the 2-dominating set. We define the state of a column of size m as the vector whose entries are the states of the cells in that column: the index 0 refers to the upper cell of the column.

We define the set $\mathcal{F}$ of the *first states* as the set of column states which can be the first known column. $\mathcal{F}$ is the set of $V \in \{0,1,2\}^n$ such that ($V[-1]$ and $V[n]$ are ignored):

- $V[i] = 2$ if and only if exactly one among $V[i-1]$ and $V[i+1]$ equals 0;
- $V[i] = 1$ if and only if both $V[i-1]$ and $V[i+1]$ are equal to 0.

We say that a column state b is *compatible* with a column state a if b can follow a, that is:

- if $a[i] = 2$ then $b[i] = 0$ (the line i of state a needed one neighbouring stone to be 2-dominated);
- if $b[i] = 1$ then at least two among $a[i]$, $b[i-1]$ and $b[i+1]$ are 0;
- if $b[i] = 2$ then exactly one among $a[i]$, $b[i-1]$ and $b[i+1]$ is 0.

Let M be the transfer matrix such that $M[a, b]$ is $+\infty$ if state b is not compatible with state a, or $|b|_0$ otherwise. Let S_1 be the vector such that $S_1[s]$ is $+\infty$ is $s \notin \mathcal{F}$, or $|s|_0$ otherwise. One can easily see that if $S_m = M^{m-1}S_1$, then we have $\gamma_2(n, m) = \min\{S_m[s] \ : \ |s|_2 = 0\}$. Indeed, we have the invariant that $S_i[s]$ is the minimum number in a grid of n lines and i columns whose last column is in state s, or $+\infty$ if this is not possible. Hence we get an algorithm computing $\gamma_2(n, m)$. This algorithm is linear in m, but exponential in n. Nevertheless, we are able to compute values for $n = 15$ in a couple of hours on a i7 core.

To determine $\gamma_2(n, m)$ for an arbitrary m we use:

Fact 1. *M is a primitive matrix, that is there is a $k \in \mathbb{N}$ such that for all i, j $M^k[i, j] < +\infty$.*

Lemma 1. *If M is primitive, then there is $m_0, k, p \in \mathbb{N}$ such that $M^{m+k} = M^m + p$ for all $m \geq m_0$.*

This relation immediately gives S_m and $\gamma_2(n, m)$ for all m. With this method we find for instance that $\forall m \geq 31, \gamma_2(7, m) = \gamma_2(7, m-18) + 53$. The recurrence relations we found prove Theorem 1 for $n \leq 13$. These values confirm the result of [6] and [7] for $n \leq 4$ but correct a mistake in [7] for $n = 5$. We also found that for $8 \leq n \leq 15$ and for $m \geq 12$, we have $\gamma_2(n, m) = \gamma_2(n, m-3) + n + 2$. This pattern in the recurrence allowed us to guess the values of $\gamma_2(n, m)$ for $n \geq 14$.

3 Method for computing $\gamma_2(n, m)$ when $n > 13$

In this section we use a different method, still based on a dynamic algorithm, to prove Theorem 1 when $n \geq 14$. The key concept here is the notion of loss induced by a 2-dominating set. A stone can contribute six times to the 2-domination: it replaces the two neighbouring stones needed by the cell it is on, and it reduces the number of needed adjacent stones of its four neighbours by

1. Hence, we define the *loss* of a 2-dominating set D by $\ell_{n,m}(D) = 6|D| - 2nm$: each cell needs to be dominated twice, and each stone in D contributes six times to this. We denote by $\ell_{n,m}$ the minimum of $\ell_{n,m}(D)$ among all 2-dominating sets D. We then have the following relation:

$$\gamma_2(n,m) = \frac{\ell_{n,m} + 2nm}{6} \tag{1}$$

We first compute a lower bound on $\ell_{n,m}$. This gives a lower bound for $\gamma_2(n,m)$ for $m \geq n \geq 14$ which matches the value of Theorem 1. We give afterwards a construction of a 2-dominating set of this size, hence proving the theorem.

We compute the minimum loss over a border of size 6, that is the minimum loss over the set X of cells at distance at most 5 from the cells of the first and last rows and columns of the grid. The algorithm used here need an extra **state 3**, meaning that no known neighbour of that cell contains a stone. To compute a minimum loss over the border of the grid, we split it into four *corners*: the top left subgrid of size 6×6, and the other three corners (top right, bottom left and bottom right). We call the four components of X minus the corners the *side band*.

We begin by computing the minimum loss on the bottom side band. To achieve this, we create a transfer matrix T such that $T[a,b]$ is the loss induced by putting state b after state a, or $+\infty$ if b is not compatible with a. Note that only the first cell of a column may be in state 3 here. Now $T^k[a,b]$ corresponds to the minimum loss of a band of k columns, whose first state is a and last state is b, without counting the loss of a on itself.

Fact 2. *The matrix T is primitive.*

We continue by computing the minimum loss on a corner, such that we can combine the loss on the bottom side band with the loss on the bottom right corner. We create a matrix Q such that $Q[a,b]$ is the minimum loss if the column of size 6 preceding the corner is already in state a and the line succeeding this corner is in state b. The losses included are the one from the state of the corner column and by b on themselves and each other, the loss by a on the state of the corner, and vice-versa.

Now $T^kQ[a,b]$ is the minimum loss induced on the side band of size k and on the corner, beginning with state a and leaving the first column of the right side band in state b. Therefore $(T^{n-13}QT^{m-13}Q)^2[a,a]$ is the minimum loss over the border of size 6 of the grid $G_{n\times m}$, assuming and setting the first column of the bottom side band to be in state a. We deduce a lower bound on the minimum loss of a 2-dominating set: $\ell_{n,m} \geq \ell_B(n,m) = \mathrm{Tr}((T^{n-13}QT^{m-13}Q)^2)$. We can again find the values of $\ell_{n,m}(B)$ for any n and m: Fact 2 and Lemma 1 guarantee that we can find a recurrence relation ($T^{k+3} = T^k + 2$ for $k \geq 15$). Using this method, we are able to show that $\gamma_2(n,m) \geq \left\lfloor \frac{(n+2)(m+2)}{3} \right\rfloor - 6$ for $14 \leq n \leq m$.

To show that this bound is sharp, we construct 2-dominating sets of the right size. We use one optimal solution for the infinite grid $\mathbb{Z}^2$: the set $D = \{(i,j) : i + j \equiv 0 \bmod 3\}$, or one of its translations. Then we project this 2-dominating set for $\mathbb{Z}^2$ into our finite $n\times m$ grid. After this, we apply a small set of rules to obtain a 2-dominating set for $G_{n,m}$. There are two rules to modify the corners of the grid, according to the pattern of this corner. Finally, we add vertices on the border such that the resulting set is 2-dominating. One can show that for $14 \leq n \leq m$ one of the resulting 2-dominating sets has the right size. One example for the grid $G_{18\times30}$ is shown in Figure 1. The first rule is used in the top left corner and the second rule is used in the top right and bottom left corner. No modification needs to be done in the bottom right corner.

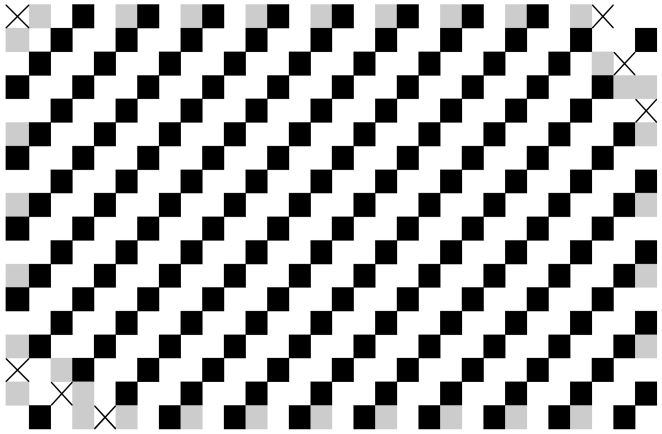


Figure 1: An optimal 2-dominating set D on a 18×30 grid. D is the set of gray or black cells. The black cells and the crossed cells are the projection of a minimal $\mathbb{Z}^2$ 2-dominating set on the grid.

4 Perspectives

We also adapted the method to solve the Roman domination on grids. A graph is said to be Roman dominated if each vertex has one or two stones, or is neighbour with a vertex containing two stones. We give the general case on grids:

$$\forall\, m \geq n \geq 10, \quad \gamma_R(n,m) = \left\lfloor \frac{2(n+1)(m+1)-2}{5} \right\rfloor - \delta_{nm \bmod 25=16}$$

We believe that the method could be extended to other types of domination in grids, such that the distance-k domination, in which it is required that every vertex not in D is at distance at most k from D. It would be interesting to know which conditions are necessary for this method to work for other graph parameters. It is strongly related to the problem of tiling uniquely the plane with a specific shape. This would explain for example why this method does not work for the total domination on grids, whose associated shape tile the plane in several different ways.

References

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