The dimension of posets with excluded minors in cover graphs

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Abstract

A fan is a graph obtained from a path by adding an extra vertex adjacent to all vertices on the path. We give a qualitative structure theorem for graphs excluding a fan as a minor. This is inspired by a recent result by Ding that gives an approximate description of graphs excluding $K_{2,n}$ as a minor. Next, we use both characterizations to show that the dimension of a poset is bounded in terms of the size of a largest $K_{2,n}$ or a fan (graph) which is a minor of the cover graph. This is a step towards characterization of minor-closed graph classes such that posets with cover graphs from such a class has bounded dimension.

In this paper we are concerned with finite posets. An important measure of a poset complexity is its dimension. The *dimension* of a poset is the least integer d, such that there are d linear extensions of P, whose intersection is P. Equivalently the dimension of P is the least integer d, such that $\mathbb{R}^d$ with the product order contains a subposet isomorphic to P. The dimension plays a similar role in poset theory, as the chromatic number does in graph theory.

For any two points x and y of a poset P, we say that x *covers* y in P if $x > y$ in P, but there is no point z such that $x > z > y$ in P. The *cover graph* of P is a graph, whose vertex set is the ground set of P, and two points are connected with an edge, if one of the points covers the other in P. A *Hasse diagram* of a poset P is a drawing of a cover graph of a poset P on the plane with x higher in the plane than y whenever $x > y$ in P. Figure 1 illustrates a diagram of a poset in well-known Kelly's construction of posets with planar cover graphs and having arbitrarily large dimension.

Connections between dimension of posets and structure of their cover graphs have been studied over years, leading to many interesting results. There are many classes of graphs known to have the property that the posets with cover graphs in the class have dimension upper bounded by a function of their height. Joret et al. [5] showed that any class of graphs with bounded expansion has that property. There are not many graph classes with the property that all posets with the cover graphs in the class have dimension bounded by an absolute constant. Trotter and Moore [6] showed that posets, whose cover graphs are trees have dimension at most 3. Felsner, Trotter and Wiechert [2] proved that if the cover graph of a poset is outerplanar, then the dimension of the

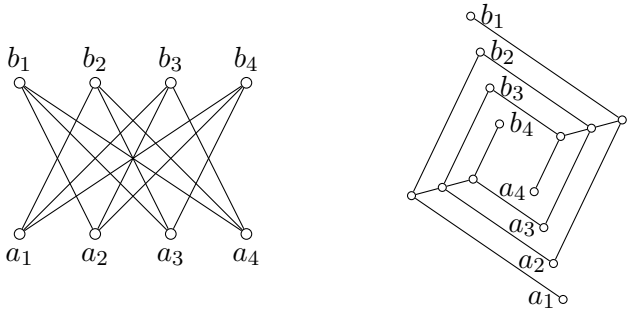


Figure 1: Kelly's construction of order 4 (right). The subposet induced by the elements a_i and b_i is a so-called standard example (left). Standard examples are known to have arbitrarily large dimension.

poset is at most 4. Later, Joret et al. [4] showed an analogous bound for a larger minor-closed class of graphs, but with worse constant, namely they showed that posets with K_4-minor-free cover graph have dimension at most 1276. On the other hand, Kelly's construction gives posets, whose cover graph have no K_5-minors, but the dimension is arbitrarily large.

We managed to show two infinite families of graphs such that for any graph H belonging to one of these families, the posets whose cover graphs exclude an H-minors have dimension bounded by the same constant (depending on H). The first family is the family of the so-called fans. For an integer $n \geq 2$, a *fan* F_n is a graph obtained from a path on n vertices by adding an extra vertex adjacent to all vertices on the path. The second family is the family of the complete bipartite graphs of form $K_{2,n}$.

Theorem 1. *For every integer $n \geq 2$ there exists an integer d such that any poset whose cover graph does not contain a fan F_n as a minor has dimension at most d.*

Theorem 2. *For every integer $n \geq 2$ there exists an integer d such that any poset whose cover graph does not contain a $K_{2,n}$-minor has dimension at most d.*

We note that Felsner et al. [3] announced that they proved the Theorem 1, but they decided not to publish this result because of the complexity of their proof.

Our proofs rely on structural properties of the graphs excluding a $K_{2,n}$-minor or an F_n-minor. In [1] Ding gave an approximate characterization of graphs which do not contain large $K_{2,n}$-minors. The characterization is quite complex, but applying the strategy from Ding's paper, we obtained a simple characterization of graphs without large F_n-minor. Ding's characterization of graphs without large $K_{2,n}$-minors and our characterization of graphs without large F_n-minors are examples of "qualitative" structure theorems of a similar flavor as Robertson-Seymour Grid Theorem, or Erdős-Pósa Theorem. The former theorem loosely speaking says that a graph contains a large grid-minor if and only if its tree-width is large, and the latter says that a graph does not contain many disjoint cycles if and only if there exists small set of vertices whose deletion leaves the graph acyclic.

In the formulation of our qualitative structure theorem for graphs without large F_n-minor we allow the graphs to have multiple edges but no loops. Let $t \geq 0$ and let G_1, G_2^1, ..., G_2^t be graphs such that for $i \in \{1, \ldots, t\}$ the intersection $G_1 \cap G_2^i$ is a 2-clique shared by the graph G_1 and G_2^i, but the graphs are otherwise disjoint, that is for any pair of distinct indices $i, j \in \{1, \ldots, t\}$ we have $G_2^i \cap G_2^j \subseteq G_1$. Then the graph

$$G = \left(G_1 \cup \bigcup_{i=1}^{t} G_2^i \right) - \bigcup_{i=1}^{t} E(G_1 \cap G_2^i)$$

is the *one-to-many 2-clique sum* of the graph G_1 with the graphs G_2^1, ..., G_2^t. The class of graphs that can be obtained by one-to-many 2-clique-summing a graph from a class $\mathcal{G}_1$ with graphs from a class $\mathcal{G}_2$ is denoted by $\mathcal{G}_1 \oplus \mathcal{G}_2$. We define inductively $\mathcal{G}^1 = \mathcal{G}$ and $\mathcal{G}^{k+1} = \mathcal{G}^k \oplus \mathcal{G}$.

For an integer $m \geq 1$ let $\mathcal{D}_m$ denote the class of graphs that can be obtained from a graph of tree-depth at most m by subdividing the edges any number of times.

Theorem 3. *A minor-closed class of graphs $\mathcal{G}$ does not contain all fans F_n if and only if there exists an integer $m \geq 1$ such that all 2-connected graphs from $\mathcal{G}$ are contained in $(\mathcal{D}_m)^m$.*

All fans are 2-connected, so it follows from Theorem 3 that a graph does not contain a large F_n-minor if and only each block of the graph is in $(\mathcal{D}_m)^m$ for some small m.

In [1] Ding gives a characterization of graphs without large $K_{2,n}$-minors, which can be formulated in a similar way[1] to Theorem 3, but with an important difference. For every integer $m \geq 2$ a class $\mathcal{A}_m$ of labeled graphs can be defined, where each edge of each graph in the class has a label determining, whether the edge can be used for 2-clique-summing. When one-to-many 2-clique-summing a labeled graph G_1 with labeled graphs $G_2^1, \ldots, G_2^t$, we require that for every $i \in \{1, \ldots, t\}$ the edge shared by the graphs G_1 and G_2^i allows 2-clique-summing in both graphs G_1 and G_2^i. For a labeled class of graphs $\mathcal{G}$, we define the classes $\mathcal{G}^k$ as above, but with the requirement, that all 2-clique-summing agree with the labels of the edges. The definition of the classes of labeled graphs $\mathcal{A}_m$ is quite complex and we do not include it in this paper.

Theorem 4 (Ding, [1])**.** *A minor-closed class of graphs does not contain all graphs $K_{2,n}$ if and only if there exists an integer m such that all 2-connected graphs in $\mathcal{G}$ are contained in some $(\mathcal{A}_m)^m$.*

Theorem 3 and Theorem 4 are similar, so it is no surprise that Theorem 1 and Theorem 2 have similar proofs. We proceed in two steps. First we decompose the cover graph of a poset P into graphs with simple structure (that is belonging to $\mathcal{D}_m$ or $\mathcal{A}_m$) and show that the dimension of the poset is bounded near every graph H in the decomposition. Secondly we show that from the fact, that the dimension is bounded locally, and we only performed one-to-many 2-clique-summing at most m times, it follows that the dimension of the whole poset P is bounded. The main difficulty in the second step is that in one-to-many 2-clique-summing the number of graphs we 2-clique-sum with is unbounded.

The first step is a technical part that needs to be proved separately for fans and $K_{2,n}$s, but the second step is abstracted to a lemma, which we believe can be useful in proving that excluding other minors than fans or $K_{2,n}$s also give bounded dimension.

It is also possible, that the lemma will find other applications, as it can be easily generalized to the graphs obtained in a similar way, but with one-to-many k-clique-summing for any fixed k.

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References

[1] G. Ding. Graphs without large $K_{2,n}$-minors. 2017. arXiv: 1702.01355.

[2] S. Felsner, W.T. Trotter, V. Wiechert. The dimension of posets with planar cover graphs. *Graphs Combin.* 31(4):927–939, 2015.

[3] S. Felsner, G. Joret, W.T. Trotter, V. Wiechert. private communication.

[4] G. Joret, P. Micek, W.T. Trotter, R. Wang, V. Wiechert. On the Dimension of Posets with Cover Graphs of Treewidth 2. *Order*, 34(2):185–234, 2016.

[5] G. Joret, P. Micek, V. Wiechert. Sparsity and dimension. *Combinatorica* (2017) in press.

[6] W.T. Trotter, J.I. Moore. The dimension of planar posets *J. Combinatorial Theory Ser. B* 22(1):54–67, 1977.

[1] The formulation in the original paper is different, and we decided to reformulate it so that it is compatible with our results.