The minimum number of triangles in graphs of given order and size

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Abstract

Rademacher (1941; unpublished) asked for the minimum number of triangles in a graph of given order and size, and solved this problem in the first non-trivial case when the size is by 1 larger than the Turán number of the triangle. This problem was revived by Erdős in the 1950s and has attracted much attention since then. In a major breakthrough, Razborov solved it asymptotically in 2008. We provide an exact solution for all large graphs whose edge density is bounded away from 1, confirming a conjecture of Lovász and Simonovits from 1975 in this range.

1 Overview of some previous results

Let K_r denote the complete graph with r vertices. Let $g_r(n,e)$ be the minimum of $g_r(G)$, the number of copies of K_r in a graph G, over all graphs with n vertices and e edges. Turán's [21] famous result (see also Mantel [10]) states that the largest K_r-free graph is the complete balanced $(r-1)$-partite graph $T_{r-1}(n)$. In other words, $g_r(n,e)=0$ if and only if $e \le t_{r-1}(n)$, where $t_k(n)$ denotes the number of edges in $T_k(n)$.

Here we concentrate on the case $r=3$, when we minimise the number of triangles. Rademacher (1941; unpublished) showed that $g_3(n, t_2(n)+1) = \lfloor n/2 \rfloor$. The triangle minimisation problem was revived by Erdős [2] and attracted much attention since then. It is now widely called the *Erdős-Rademacher problem*.

It is not hard to show using the triangle-counting method of Goodman [6] that $g_3(n,t_k(n)) = g_3(T_k(n))$ and many early results concentrated on the case when e is slightly above the Turán number $t_k(n)$, see [2, 3, 8, 9, 13, 14]. The strongest result (for $n \to \infty$) in this direction was proved by Lovász and Simonovits [9] and is as follows.

Define $\mathcal{H}(n,e)$ to be the family of graphs with n vertices and e edges that can be obtained from a complete multipartite graph by adding a triangle-free graph into one part, and let

$$h(n,e) := \min\{g_3(H) : H \in \mathcal{H}(n,e)\}. \tag{1}$$

Lovász and Simonovits [9] proved that for every integer $k \ge 3$ there exists $\alpha = \alpha(k) > 0$ such that $g_3(n,e) = h(n,e)$ for all positive integers (n,e) with $t_k(n) \le e \le t_k(n) + \alpha n^2$. In fact they were able to characterise the extremal graphs in this range.

Also, Lovász and Simonovits [8, 9] made the following bold conjecture.

Conjecture 1 (Lovász and Simonovits [8, 9])**.** *There is n_0 such that for all $n \ge n_0$ and $0 \le e \le \binom{n}{2}$, we have $g_3(n,e) = h(n,e)$.*

In order to state some of the following results, we define for $\lambda \in [0,1]$

$$g_3(\lambda) := \lim_{n\to\infty} \frac{g_3(n,e(n))}{\binom{n}{3}}, \quad \text{where } e(n) = (\lambda + o(1))\binom{n}{2},\ 0 \le e(n) \le \binom{n}{2}. \tag{2}$$

It is easy to see that the limit exists and does not depend on the choice of the function $e(n)$.

General lower bounds on $g_3(\lambda)$ were proved by Moon and Moser [11], Nordhaus and Stewart [15], and Bollobás [1]. Then Fisher [5] determined $g_3(\lambda)$ in the range $1/2 \le \lambda \le 2/3$. In a major breakthrough, Razborov [17, 18] used his newly developed theory of *flag algebras* to determine $g_3(\lambda)$ for all $\lambda \in [0,1]$ (and this result was extended to $r = 4$ by Nikiforov [12] and to arbitrary r by Reiher [19]).

Razborov's result is consistent with Conjecture 1. It is routine to show that the value of $h(n,e)$ is attained within additive error $o(n^3)$ by a complete multipartite graph consisting of k (larger) parts of size $(c+o(1))n$ and one part of size $(1-kc+o(1))n$, where the integer k satisfies $1-\frac{1}{k} \le \lambda < 1-\frac{1}{k+1}$ and c is uniquely determined by $\binom{k}{2}c^2 + kc(1-kc) = \lambda/2$ and $c \ge 1-kc$. The corresponding upper bound on $g_3(\lambda)$ has a scalloped shape: it is a concave function between the (convexly placed) special points $g_3(1-\frac{1}{k}) = \frac{(k-1)(k-2)}{k^2}$, $k \in \mathbb{N}$, that correspond to the Turán graph $T_k(n)$. Thus, by Razborov's result, this upper bound on $g_3(\lambda)$ is equality for every $\lambda \in [0,1]$.

More recently, Pikhurko and Razborov [16] proved a "stability" version: every almost $g_3(n,e)$-extremal graph is $o(n^2)$ close in the edit distance to some graph in $\mathcal{H}(n,e)$.

2 Our contribution

First, we observe that the problem of determining $h(n,e)$, i.e. finding the minimum in the right-hand side of (1), can be solved exactly for every (n,e).

Proposition 1 (Liu, Pikhurko and Staden [7]). *For every $0 \le e \le \binom{n}{2}$, the value of $h(n,e)$ is attained by taking a complete $(k+1)$-partite graph $K[A_1,\dots,A_{k+1}]$ with $|A_1| \ge \dots \ge |A_{k+1}|$ and removing a (possibly empty) star between A_k and A_{k+1}.*

Note that each graph H from the above proposition can be represented as adding a triangle-free graph to the complete k-partite graph $K[A_1,\dots,A_{k-1},B]$, where $B = A_k \cup A_{k+1}$; so H belongs to the family $\mathcal{H}(n,e)$. In fact, the number of parts and their exact sizes in Proposition 1 can be explicitly stated, see [7, Proposition 1.8]. Namely, one takes the smallest possible number $k+1$ of parts, then minimises the size of the smallest part, and then makes all other parts as equal as possible. An interesting consequence of Proposition 1 that has not been observed before is that if Conjecture 1 is true, then its conclusion is in fact true for **all** $n \ge 1$, see [7, Lemma 10.1].

Our main result proves Conjecture 1 if the edge density is bounded away from 1:

Theorem 1 (Liu, Pikhurko and Staden [7]). *For all $\varepsilon > 0$, there exists $n_0 > 0$ such that for all positive integers $n \ge n_0$ and $e \le \binom{n}{2} - \varepsilon n^2$, we have $g_3(n,e) = h(n,e)$.*

Furthermore, in [7, Theorem 1.9] we give an explicit description of all extremal graphs in this range. Unfortunately, this description is quite technical to state (in fact, every $g_3(n,e)$-extremal graph comes from one of three related but different graph families $\mathcal{H}^*_i(n,e)$, $i = 0,1,2$, with each family needed for this result).

The asymptotic proofs of Fisher, Nikiforov, Pikhurko-Razborov, Razborov and Reiher all use algeraic and/or analytic methods. These techniques do not seem to be helpful for the exact problem, and indeed our proof of Theorem 1 uses purely combinatorial methods. At its heart, our proof is based on the well-known stability method that goes back to Erdős and Simonovits [4, 20].

Informally speaking, the proof proceeds as follows. We assume that the desired inequality $g_3(n,e) \ge h(n,e)$ is false for some (n,e) in our range and take a *worst counterexample* G of order

n, that is, a graph that maximises $h(n, e(G)) - g_3(G)$. By Pikhurko and Razborov's result [16] we know that G is structurally close to some H in $\mathcal{H}(n, e)$, where $e := e(G)$.

The standard strategy here would be to analyse G and show by means of local transformations (such as, for example, deleting one vertex and cloning another) that G has a simple structure and then derive a contradiction. However, a significant obstacle is the fact that there is a large family of conjectured extremal graphs: namely, when we add a triangle-free graph F of a given size into one part of a complete multipartite graph P, then there are usually many choices of F, all of them giving the same number of triangles in the final graph. In fact, for some (n, e), an extremal graph can be obtained by adding extra edges into more than one part of P. Also, for some other (n, e), there are extremal graphs obtained from a complete multipartite graph by removing edges spanning more than two parts.

Our proof first obtains as much information as possible about G by exploiting local transformations. However, this does not seem enough yet to get a contradiction. Then, roughly speaking, we make global changes that convert G into a $(k+1)$-partite graph H of the same order n and the same size e, where k satisfies $t_k(n) < e \le t_{k+1}(n)$. While we cannot guarantee that $g_3(H) \le g_3(G)$, we can show that $g_3(H) - g_3(G)$ is small and that the part sizes of H deviate from the optimal ones to a degree that allows us to prove that $g_3(H) - h(n, e) > g_3(H) - g_3(G)$. This gives the desired contradiction $g_3(G) > h(n, e)$. Unfortunately, the rigorous proof is very long (around 85 pages) and technical (in particular, requiring a number of different types of transformations). Its full details can be found in [7].

The main remaining open problems are to settle Conjecture 1 in the whole range and to prove exact results for $g_r(n, e)$ when $r \ge 4$.

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