Excluded Minors for Embeddings of Metric Graphs in ℓ_∞^n-Spaces

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Abstract

A *metric graph* is a pair (G, d), where $G = (V, E)$ is a graph and $d : E \to \mathbb{R}_{\geq 0}$ is a distance function on E. An *isometric embedding* of the metric graph (G, d) in $\ell_\infty^n = (\mathbb{R}^n, d_\infty)$ is a set of vectors $\{q_v\}_{v \in V} \subseteq \mathbb{R}^n$ such that $d_\infty(q_v, q_w) = d(vw)$ for all $vw \in E$. The graph parameter $f_\infty(G)$ is the least integer n such that there exists an isometric embedding of (G, d) in ℓ_∞^n for all distance functions d.

The property $f_\infty(G) \leq n$ is closed under taking minors. By the Graph Minor Theorem of Robertson and Seymour, there exists a finite list of excluded minors for the property $f_\infty(G) \leq n$.

We present four graph families of unbounded f_∞-value and show that they are unavoidable in *any* graph with large f_∞-value. This can be viewed as an approximate characterization of graphs with large f_∞-value in the same vein as the grid theorem for graphs of large treewidth.

1 Introduction

In this paper, we consider isometric embeddings of graphs in metric spaces. Recall that a *metric space* (X, d) consists of a *set of points* X and a *metric* $d : X \times X \to \mathbb{R}_{\geq 0}$. That is, for all $x, y, z \in X$, (i) $d(x, y) = d(y, x)$, (ii) $d(x, y) = 0$ if and only if $x = y$, and (iii) $d(x, y) \leq d(x, z) + d(z, y)$. We only consider the metric spaces $\ell_p^n = (\mathbb{R}^n, d_p)$. Recall that $\|x\|_p = (\sum_{i=1}^n |x|^p)^{1/p}$ if $p \in [1, \infty)$ and $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$. We set $d_p(x, y) = \|y - x\|_p$ for all $p \in [1, \infty]$.

Comparing different metric spaces has been the subject of various theories in mathematics. One way to do so is by means of *isometric embeddings*, i.e. functions $\phi : X \to X'$ such that $d(x, y) = d'(\phi(x), \phi(y))$ for all $x, y \in X$. As these are quite restrictive, other approaches have been developped. For instance, in the theory of distortion we require that the distances are preserved up to a given factor. Bourgain [3] has shown that any finite n-point metric space can be embedded into an $\ell_p^{O(\log^2 n)}$ space with $O(\log n)$ distortion.

Another point of view is to require only a subset of distances to be preserved, which leads to a graph theoretical approach which we will pursue in this paper. All graphs we discuss are finite and do not contain loops or parallel edges. A graph H is a *minor* of a graph G if H can be obtained from G by deleting edges or vertices, or contracting edges in any order. When taking minors we remove parallel edges and loops.

A *metric graph* (G, d) is a pair consting of a graph $G = (V, E)$ and a *distance function* $d : E \to \mathbb{R}_{\geq 0}$ satisfying $d(vw) \leq d(P) = \sum_{i=1}^m d(v_{i-1}v_i)$ for all $vw \in E(G)$ and all paths $P = v_0 v_1 \dots v_k$ with $v_0 = v$ and $v_k = w$. An *isometric embedding* of a metric graph in ℓ_p^n is a set of vectors $\{q_v\}_{v \in V} \subseteq \mathbb{R}^n$ such that $d_p(q_v, q_w) = d(vw)$ for all edges $vw \in E$.

For each $p \in [1, \infty]$ and graph $G = (V, E)$, a distance function $d : E \to \mathbb{R}_{\geq 0}$ is *ℓ_p-realizable* if it comes from an isometric embedding in ℓ_p^m (m can be arbitrarily large). If d is ℓ_p-realizable, we define the parameter $f_p(G, d)$ to be the least integer n such that (G, d) can be embedded in ℓ_p^n. Furthermore we set $f_p(G) = \max_d f_p(G, d)$, where the maximum is over all ℓ_p-realizable distance functions. By the conic version of Carathéodory's theorem this parameter is always finite.

It is easy to see that a minor H of G satisfies $f_p(H) \leq f_p(G)$ for all $p \in [1, \infty]$. Hence the property $f_p(G) \leq n$ is closed under taking minors. By the Graph Minor Theorem of Robertson and Seymour, for each n, there exists a finite list of *excluded minors* for the property $f_p(G) \leq n$. Formally, an excluded minor for $f_p(G) \leq n$ is a graph H such that $f_p(H) > n$ and every proper minor H' of H satisfies $f_p(H') \leq n$.

The complete lists of excluded minors are known in the Eucledian case ℓ_2^n for the dimensions $n = 1, 2, 3$. Belk and Connelly [1, 2] have shown that $\{K_3\}$, $\{K_4\}$, $\{K_5, K_{2,2,2}\}$ are the respective sets of excluded minors. Furthermore, it is easy to see that the only excluded minor for $f_p(G) \leq 1$ is K_3 for all $p \in [1, \infty]$ by setting the same distance on each edge.

Fiorini, Joret, Huynh and Varvitsiotis [4] studied ℓ_∞^2. The excluded minors are W_4, the wheel on 5 vertices, and the graph $K_4 +_e K_4$, which is obtained by gluing two K_4-graphs on a common edge and deleting that edge.

We will present some general results about ℓ_∞^n-spaces with $n \geq 3$.

2 Alternative views of f_∞

We now introduce alternative views of the parameter f_∞ using potentials and a variant of edge coloring.

Let D be a directed edge-weighted graph and $l : A(D) \to \mathbb{R}$. A *potential on* (D, l) is a function $p : V(D) \to \mathbb{R}$ such that $p(w) - p(v) \leq l(a)$ for all $a = (v, w) \in A(D)$. Recall that in ℓ_∞^n we define $d_\infty(x, y) = \max_{i=1,\ldots,n} |x_i - y_i|$. Hence $|x_i - y_i| \leq d_\infty(x, y)$ for all $i = 1, \ldots, n$ and there exists some index j for which $|x_j - y_j| = d_\infty(x, y)$.

Let (D, l) be the directed edge-weighted graph obtained from (G, d) by bidirecting all edges and setting $l(\overrightarrow{vw}) = l(\overrightarrow{wv}) = d(vw)$ for all $vw \in E(G)$. Note that $p : V(D) \to \mathbb{R}$ is a potential if and only if $|p(w) - p(v)| \leq d(vw)$ for all $vw \in E(G)$. We say that an edge vw is *tight* if $|p(w) - p(v)| = d(vw)$.

Notice that in ℓ_∞^n-spaces, a set $\{q_v\}_{v \in V}$ is an embedding of some (G, d) if and only if there exist n potentials $p_i : V \to \mathbb{R}$ on (D, l) such that each edge is tight in some potential. We can define p_i such that $p_i(v)$ corresponds to the i-th coordinate of q_v in $\mathbb{R}^n$. Hence $f_\infty(G)$ corresponds to the least integer n such that for each metric graph (G, d) there exists a family of n potentials on (D, l) such that each edge of G is tight in at least one potential.

Given a set of edges $F \subseteq E(G)$ we say that F *is feasible* if there exists a potential $p : V \to \mathbb{R}$ on (D, l) such that $|p(w) - p(v)| = d(vw)$ for all $vw \in F$, otherwise F is *unfeasible*. If F is feasible, then there exists a *feasible orientation* $\overrightarrow{F}$ of the edges of F such that $p(w) - p(v) = -d(vw)$ if $\overrightarrow{vw} \in \overrightarrow{F}$ and $p(w) - p(v) = d(vw)$ otherwise. Notice that all edges which are tight for some fixed p_i form a feasible set, and each feasible set F gives rise to a potential p such that the edges of F are tight. Hence $f_\infty(G)$ is the least integer n such that for each (G, d) there exists a collection of n feasible sets $F_1, \ldots, F_n$ such that $E(G) \subseteq \bigcup_{i=1}^n F_i$.

An *n-coloring* of the edges of G is a partition of $E(G)$ into sets $F_1, \ldots, F_n$ called *color classes*. If all color classes are feasible sets, we say that the coloring is *feasible*.

Given a feasible orientation $\overrightarrow{F}$, we can modify the length function $l = l(\overrightarrow{F})$ on D such that

$$l(\overrightarrow{vw}) = \begin{cases} d(vw) & \text{if } \overrightarrow{vw} \notin \overrightarrow{F} \\ -d(vw) & \text{if } \overrightarrow{vw} \in \overrightarrow{F} \end{cases}.$$

By a well-known result characterizing the existence of a potential, the weighted digraph $(D, l(\overrightarrow{F}))$ admits a potential if and only if it does not contain a negative weight directed cycle. We say that

the color class F does not have a negative cycle if there exists an orientation $\overrightarrow{F}$ of F, such that $(D, l(\overrightarrow{F}))$ has no negative cycle.

Given an n-coloring of G, we can define $(D, l(\overrightarrow{F_i}))$ for each color class F_i. Hence, $f_\infty(G) = n$ if and only if for all (G, d) there exists a feasible n-coloring of $E(G)$, i.e. such that each color class has no negative cycle.

3 New results

For all $n \geq 2$, it is known that the excluded minors for $f_\infty(G) \leq n$ have minimum degree at least 3. Fiorini et al. [4] proved that deleting a degree-2 vertex v and adding a new edge between the neighbours of v (if there was none) does not change $f_\infty(G)$.

We now mention a useful lemma about gluing two graphs on an edge. We write $G = G_1 \oplus_{vw} G_2$ if $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v, w\}$ and $vw \in E(G_1), E(G_2)$. Also, we denote by $G_1 +_{vw} G_2$ the graph $G_1 \oplus_{vw} G_2$ minus the edge vw.

Lemma 1. *Let G_1, G_2 be two graphs with $V(G_1) \cap V(G_2) = \{v, w\}$ and $vw \in E(G_1), E(G_2)$. Then*

$$f_\infty(G_1 +_{vw} G_2) \leq f_\infty(G_1 \oplus_{vw} G_2) \leq f_\infty(G_1) + f_\infty(G_2) - 1.$$

The first inequality is trivial since $G_1 +_{vw} G_2$ is a minor of $G_1 \oplus_{vw} G_2$. The key idea for the second inequality is to see that we can color G_1 and G_2 independently using $f_\infty(G_1) + f_\infty(G_2)$ colors in total (the color classes of each G_i are still feasible in the whole graph). After this, we can merge the two color classes which are used for vw in G_1 and G_2 respectively. The new color class is feasible because we can add $\overrightarrow{vw}, \overrightarrow{wv}$ to any cycle C using edges of both G_1 and G_2 and decompose C into two cycles C_1, C_2 which are respectively contained in $E(G_1), E(G_2)$.

Another important observation is that we can sometimes find sets of edges of G that cannot pairwise share a color for some distance function d. For two edges v_1v_2, w_1w_2, they cannot share a color if there exist v_iw_j-paths $P_{i,j}$ such that $d(P_{1,1}) + d(P_{2,2}) > d(v_1v_2) + d(w_1w_2)$ and $d(P_{1,2}) + d(P_{1,2}) > d(v_1v_2) + d(w_1w_2)$. This gives a lower bound on $f_\infty(G)$.

Combining these two results, we can provide upper and lower bounds for some graph families. The *n-K_4-star* is obtained from n K_4 graphs by gluing them on one common edge and deleting that edge. The *n-K_4-fan* is obtained by gluing n K_4-graphs on adjacent edges which share a common vertex and deleting these edges. The *n-K_4-path* is obtained from n K_4 graphs by gluing them on opposite edges which are deleted afterwards. These graphs are illustrated in Figure 1.

Notice that if $n = 2$, then the three graphs are the graph $K_4 +_e K_4$ which is one of the excluded minors for $f_\infty(G) \leq 2$ identified in [4]. By applying Lemma 1 and using $f_\infty(K_4) = 2$, it follows by induction that these graphs satisfy $f_\infty(G) \leq n + 1$. There exist distance functions for each of these graphs such that there exist $n + 1$ edges that need distinct colors in every feasible coloring. The corresponding edges are highlighted in Figure 1. Hence these graphs satisfy $f_\infty(G) = n + 1$.

We will now introduce a family of 3-connected graphs with unbounded f_∞-value. The *n-necklace* N_n is the graph obtained from the n-ladder by adding the vertex v and the edges vv_1, vw_1, vv_n, vw_n. This graph is the last one in Figure 1. By adding the vertex v to the ladder we create a 3-connected graph with $f_\infty(N_n) \geq \lceil n/2 \rceil$. Indeed, it is possible to construct a distance function d on N_n such that every other of the matching edges v_iw_i requires distinct colors as shown in Figure 1.

These four families provide simple constructions of graphs with large f_∞-value. Furthermore, our main theorem shows that they are unavoidable in any graph with large f_∞-value:

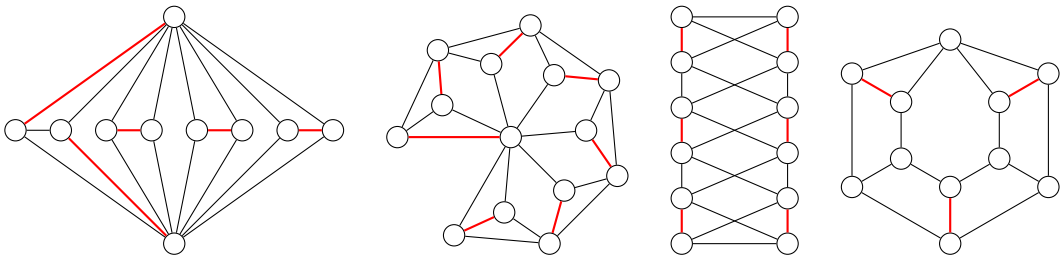


Figure 1: From left to right: 4-K_4-star, 6-K_4-fan, 5-K_4-path, 5-necklace. For each graph, there is a distance function such that the red edges must all receive distinct colors.

Theorem 1. *Let $n \geq 2$. The graphs n-K_4-star, n-K_4-fan, n-K_4-path satisfy $f_\infty(G) = n + 1$ and are excluded minors for $f_\infty(G) \leq n$. The n-necklace N_n satisfies $f_\infty(N_n) \geq \lceil n/2 \rceil$.*

On the other hand, if G is a graph containing none of n-K_4-star, n-K_4-fan, n-K_4-path and n-necklace as a minor, then $f_\infty(G)$ is bounded by a function of n.

To conclude this extended abstract, we also mention that we studied the case $n = 3$ in depth in [5]. Motivated by the problem of identifying the corresponding excluded minors, we identified a list of 38 graphs that satisfy $f_\infty(G) = 4$ and are pairwise incomparable w.r.t. the minor relation.

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