Weighted independence ratio of geometric distance graphs

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Abstract

In this work, we study the density of sets that do not contain two points at distance exactly one, especially in the case of norms induced by polytopes that tile the space by translation. In this case, Bachoc and Robins conjectured that the maximum density of a set avoiding distance 1 in $\mathbb{R}^n$ is $\frac{1}{2^n}$. We showed in previous work that this conjecture holds in dimension 2 and for certain infinite families of polytopes. This work presents an alternative approach which improves our results in dimension 3 and involves finding a weight distribution on the vertices of a graph that minimizes the ratio between the maximum weight of an independent vertex set and the total weight of the graph.

1 Context and definitions

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. A set A *avoids distance 1* if and only if $\forall (x,y) \in A^2, \|x-y\| \neq 1$. Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set. The density of A, denoted $\delta(A)$ is defined as:

$$\delta(A) = \limsup_{R\to\infty} \frac{\mathrm{Vol}(A \cap [-R,R]^n)}{\mathrm{Vol}([-R,R]^n)}$$

The supremum of the densities of Lebesgue measurable sets $A \subset \mathbb{R}^n$ avoiding distance 1 is denoted $m_1(\mathbb{R}^n, \|\cdot\|)$. This number was introduced in 1972 by Larman and Rogers [4] to study the chromatic number of $\mathbb{R}^n$.

A natural approach to build a dense set avoiding distance 1 that works for any norm starts from a packing of unit balls. Let Λ be a set such that for all distinct x and y in Λ, the unit open balls $B(x,1)$ and $B(y,1)$ do not overlap. Then the set $A = \cup_{\lambda\in\Lambda} B(\lambda, 1/2)$ of disjoint open balls of radius $1/2$ is a set avoiding 1 and its density is $\frac{\delta}{2^n}$ where n is the dimension of the space and δ is the density of the packing. This construction is illustrated in Figure 1.

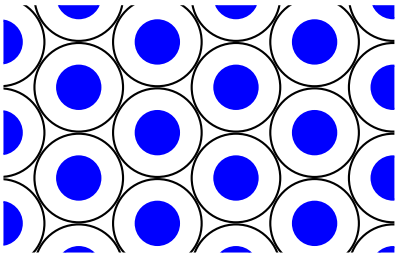


Figure 1: A set avoiding distance 1 built from a sphere packing.

The density of an optimal packing of discs of radius 1 in the euclidean plane is about 0.9069. Thus, the construction depicted in Figure 1 provides a lower bound of about $\frac{0.9069}{4} \simeq 0.2267$ for

$m_1(\mathbb{R}^2, \|\cdot\|_2)$. By refining this idea, Croft manages to build in [2] a set of density about 0.2293 which is the best lower bound known for $m_1(\mathbb{R}^2, \|\cdot\|_2)$.

The best upper bound we know is due to Keleti et al. [3], who have shown that $m_1(\mathbb{R}^2) \leq 0.258795$. A conjecture by Erdős [6] states that $m_1(\mathbb{R}^2) < \frac{1}{4}$. This conjecture was generalized to higher dimensions by Moser, Larman and Rogers in [4]: for every $n \geqslant 2$, $m_1(\mathbb{R}^n) < \frac{1}{2^n}$.

This problem is also related to the Hadwiger-Nelson problem, which consists of determining the chromatic number of the plane *i.e.* the smallest number of colours required to colour each point of the plane such that no two points at distance exactly one from each other have the same colour. Given a normed vector space $(\mathbb{R}^n, \|\cdot\|)$, the *unit distance graph* $G(\mathbb{R}^n, \|\cdot\|)$ is the graph whose vertex set is $\mathbb{R}^n$, and which has an edge between each pair of points at distance one from each other. The Hadwiger-Nelson problem consists of determining the chromatic number of this graph and our problem consists of determining the highest density of a measurable independent set.

2 Polytope norms and Bachoc-Robins conjecture

Every convex symmetric polytope $\mathcal{P} \subset \mathbb{R}^n$ centered at 0 and with a non empty interior is the unit ball for a given norm. This norm is denoted by $\|\cdot\|_{\mathcal{P}}$ and is given by the following formula:

$$\|x\|_{\mathcal{P}} = \inf\{\lambda \in \mathbb{R}^+ : x \in \lambda\mathcal{P}\}$$

A polytope $\mathcal{P}$ *tiles* $\mathbb{R}^n$ *by translation* if and only if there exists a discrete set of vectors S such that $\cup_{s\in S}(P+s) = \mathbb{R}^n$ and $\forall s, s' \in S^2$, $P+s$ and $P+s'$ have disjoint interiors.

If the unit ball associated to a norm is a polytope $\mathcal{P}$ that tiles the space by translation, the construction depicted in Figure 1 allows us to achieve a density of $\frac{1}{2^n}$, as illustrated in Figure 2.

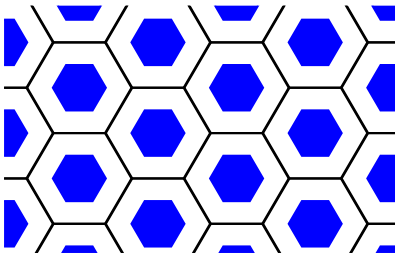


Figure 2: A set of density $1/2^n$ built from a tiling of the plane by regular hexagons.

This proves that in this case, $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) \geqslant \frac{1}{4}$. Bachoc and Robins conjectured that this construction is optimal:

Conjecture 1 (Bachoc, Robins)**.** *For all dimension n, if $\|\cdot\|$ is a norm such that the unit ball tiles $\mathbb{R}^n$ by translation, then $m_1(\mathbb{R}^n, \|\cdot\|) = \frac{1}{2^n}$.*

Note that the lower bound of Croft in the Euclidean case proves that the above construction is not always optimal when the unit ball does not tile the plane but the construction of Croft still provides a bound strictly lower than $\frac{1}{2^2}$. The conjecture of Erdös implies that because the euclidean disc does not tile the plane, the bound of $\frac{1}{4}$ cannot be achieved.

In [1], we proved that $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) = \frac{1}{2^n}$ for several families of polytopes $\mathcal{P}$, including all those who tile $\mathbb{R}^2$ by translation face-to-face (the case $n = 2$ of the conjecture of Bachoc and Robins). Those results were extended by Moustrou in [5]. All the proofs are based on the following lemma:

Lemma 1.
If $G = (V, E)$ is a discrete subgraph induced by $G(\mathbb{R}^n, \|\cdot\|)$, then $m_1(\mathbb{R}^n, ||.||) \leqslant \bar{\alpha}(G)$ where $\bar{\alpha}(G)$ denotes the independence ratio of G i.e. the maximum density of an independent set in G.

For example, if the norm is $\|\cdot\|_\infty$, then the hypercube whose vertices are the elements $\{0,1\}^n$ has cardinality 2^n and an independence number of 1. Thus, the conjecture of Bachoc-Robins is trivial in this case.

If the unit polytope is a regular hexagon, we were unable to find a finite subgraph whose independence ratio is $\frac{1}{4}$ but we were still able to achieve the bound of $\frac{1}{4}$ thanks to the infinite auxiliary graph H depicted in Figure 3 (inspired by Dmitry Shiryaev).

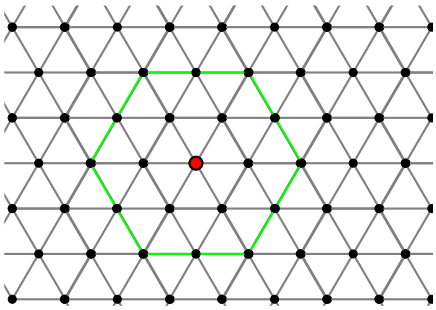


Figure 3: The auxiliary graph H. The unit polytope is depicted in green and the origin in red.

Note that the vertices at graph distance 2 from a given vertex u of H are exactly the vertices at polytope distance 1 from u. The sets of vertices that avoid polytope distance 1 are thus exactly the sets that avoid distance graph 2 *i.e.* union of cliques in H with disjoint neighbourhood. We can prove that no clique has density higher than $\frac{1}{4}$ in its neighbourhood, which proves that the independence ratio of H is $\frac{1}{4}$.

However, this approach does not extend easily in higher dimension as the property that allows us to bound the independence ratio of our auxiliary infinite graphs does not hold anymore. To overcome this difficulty, we introduce an approach based on weighted graphs.

3 Optimal weighting

If G is a finite vertex-weighted graph, we define its independence ratio $\bar{\alpha}(G)$ as the ratio between a maximum-weight independent set of G and the total weight of the graph. We can prove that Lemma 1 still holds on weighted graphs:

Lemma 2. *If G is a weighted subgraph induced by $G(\mathbb{R}^n, \|\cdot\|)$, then $m_1(\mathbb{R}^n, ||.||) \leqslant \bar{\alpha}(G)$.*

Weighted graphs allow us to achieve bounds on finite graphs that we were only able to reach on infinite unweighted graphs in [1]. Let $\mathcal{P}$ be the regular hexagon and let G be the graph depicted in Figure 4, which is a subgraph of the graph H depicted in Figure 3. There exists sets avoiding graph distance 2 of size 9 (such as the one depicted in blue), and the bound we have on $m_1(\mathbb{R}^2, ||.||_\mathcal{P})$ is therefore $\frac{9}{31} \geqslant 0.29$. However, with an appropriate weight distribution (depicted in Figure 5), the same graph now has total weight 72 and no set avoiding graph distance 2 has weight higher than 18. Hence, this finite graph proves that $m_1(\mathbb{R}^2, ||.||_\mathcal{P}) = \frac{1}{4}$.

The first problem that our work focuses on is the following: given a finite graph G (like the one depicted in Figure 4), we look for a weighting of the vertices of G that minimizes the weighted

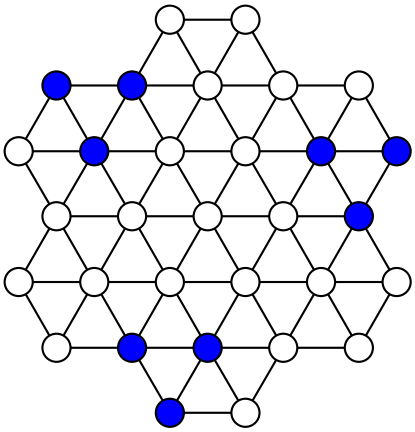


Figure 4: An independent set of ratio $> \frac{1}{4}$ in a graph G.

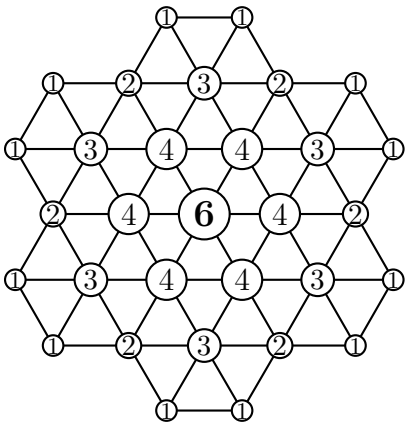


Figure 5: A weighting of G such that no independent set has a ratio $> \frac{1}{4}$.

independence ratio of G (such as the weighting depicted in Figure 5). This problem is complex since it involves two optimization problems: finding a weight distribution that minimizes the weight-ratio of a given collection of vertex sets and finding a maximum-weight independent sets for a given weighting (which is already an NP-complete problem). We look for specific properties of the induced subgraphs of unit-distance graphs for polytope norms that we can take advantage of in our algorithms and design algorithms based on integer linear programming.

Another important problem is, for a given polytope $\mathcal{P}$, to find a graph for which the previous problem can be solved as quickly as possible and that would yield an optimal independence ratio of $\frac{1}{2^n}$. For example, for the norm given by the regular hexagon, the graph depicted in Figure 4 is good because it leads to a bound of $\frac{1}{4}$, which is the case of none of its strict subgraphs and it has symmetries that help find an optimal weighting more efficiently.

Our weighted graphs allow to decrease the bounds on $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ for several polytopes $\mathcal{P}$, including the truncated octahedron, which is the only regular polytope of $\mathbb{R}^3$ for which the conjecture of Bachoc-Robins is still open ([1] [5]). Our work notably proves that the measurable chromatic number of $\mathbb{R}^3$ with a norm induced by a regular parallelohedron is 8.

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