On more variants of the Majority Problem

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Abstract

The Majority problem can be formulated as follows. Given a set $\mathcal{B}$ of N balls, each of these colored either red or green, a colorblind player, who is unable to determine the color of a ball but can decide if two balls have the same color, wants to determine a ball of the majority color within a minimum number of comparisons. This problem has been first solved by Saks and Werman. We consider the variant where the goal of the player is to determine the color of every ball. Aigner gave bounds on the minimum number of questions, denoted $Q(N, p, \leq)$, that guarantees the determination of the colors of all balls in the case where there are at most p green balls in $\mathcal{B}$. We extended the results of Aigner on exact values of $Q(N, p, \leq)$. Similarly we define $Q(N, p, =)$ for the case where there are exactly p green balls in $\mathcal{B}$. We provide upper bounds for $Q(N, p, =)$, and exact values for the first two values of p. Our results lead to several new questions.

1 Introduction

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_N\}$ be a set of N balls, each ball being either red or green. The goal is to provide the color of each of the balls by asking questions of the type "*Are b_i and b_j colored the same ?* " $(1 \leq i < j \leq N)$. We remark that $N-1$ such comparisons may lead easily to a partition of $\mathcal{B}$ into two subsets $\mathcal{B}_1$ and $\mathcal{B}_2$ that are the sets of balls of each color. If we would allow $\mathcal{B}$ to contain any number of red and green balls it would be impossible to decide the colors of the balls. Hence we will always assume in the following that $\mathcal{B}$ contains more red balls than green balls and we call such a coloring a *Red-green coloring*. Furthermore we will also consider more restricted kinds of Red-green colorings : given an integer p, a Red-green coloring of N balls will be said *p-majored* if there are at most p green balls and it will be said *p-equal* if there are exactly p green balls. The difficulty consists in determining a method minimizing the number of comparisons. Given two integers N and $p < \frac{N}{2}$ we call $(N, p, \leq)$-*identification* (respectively $(N, p, =)$-*identification*) the problem of determining for sure all the colors of N balls colored by a p-majored Red-green coloring (respectively by a p-equal Red-green coloring) and we denote by $Q(N, p, \leq)$ (respectively $Q(N, p, =)$) the minimum number of comparisons that are necessary to solve any instance of the $(N, p, \leq)$-identification problem (respectively of the $(N, p, =)$-identification problem).

We have shown that $Q(N, p, \leq)$ and $Q(N, p, =)$ share properties that are useful for proving bounds.

Let $\perp$ denotes either " $=$ " or " $\leq$ ", the two following lemmas hold.

Lemma 1. *For every two integers N and $1 \leq p < \frac{N}{2}$ we have :*

$$Q(N, p, \perp) \geq 2 + Q(N-2, p-1, \perp).$$

Lemma 2. *For every two integers N and $p < \frac{N-1}{3}$ we have :*

$$Q(N, p, \perp) \leq p + Q(N-(p+1), p, \perp).$$

2 The $(N, p, \leq)$-identification problem

In this section we present our main results on the $(N, p, \leq)$-identification problem.

Given two integers N and p such that $0 \leq p < \frac{N}{2}$, let us define

$$Q_+(N, p, \leq\ ) := N + 1 - \lfloor \frac{N+1}{p+1} \rfloor \text{ and } Q_-(N, p, \leq) := Q_+(N, p, \leq) - 1.$$

Aigner showed the following :

Theorem 1. *[1] Let N and p be integers such that $0 \leq p < \frac{N}{2}$, we have:*

- $Q_-(N, p, \leq) \leq Q(N, p, \leq) \leq Q_+(N, p, \leq)$,
- $Q(N, p, \leq) = Q_+(N, p, \leq)$ *when* $N \equiv r\ [p+1]$ *and* $r = p$ *or* 0.

The next theorem gives more cases where the exact value of $Q(N, p, \leq)$ is known and partially answers a question of Wildon (Problem 8.1 in [2]).

Theorem 2. *Let N and p be integers such that $0 \leq p < \frac{N}{2}$, we have:*
$Q(N, p, \leq) = Q_+(N, p, \leq)$ *when* $N \equiv r\ [p+1]$ *and* $r = p$ *or* $0 \leq r \leq \lfloor \frac{p}{2} \rfloor$.

Furthermore the following holds due to several properties satisfied by $Q(N, p, \leq)$.

Theorem 3. *For every nonnegative integer p exactly one of the two following statements is satisfied:*

1. *For every $N \geq 2p+1$ we have $Q(N, p, \leq) = Q_+(N, p, \leq)$.*
2. *There exist two integers $\frac{p+1}{2} \leq r_p \leq p-1$ and $N_p \geq 2p+1$ such that for every $N \geq N_p$ we have :*
   $Q(N, p, \leq) = Q_-(N, p, \leq) \iff N \equiv r\ [p+1]$ *for* $r_p \leq r \leq p-1$.

Remark that Theorem 2 shows only cases where $Q(N, p, \leq) = Q_+(N, p, \leq)$. In particular Theorem 2 proves that $Q(N, p, \leq) = Q_+(N, p, \leq)$ whenever $p = 1$ or 2, and $N \geq 2p+1$. Hence it is natural to wonder if $Q(N, p, \leq)$ is always equal to $Q_+(N, p, \leq)$. This is however not the case: Wildon [2] has checked by a computer search the values of $Q(N, p, \leq)$ for $N \leq 30$. In particular he listed all such $Q(N, p, \leq)$ that are equal to $Q_-(N, p, \leq)$. Our (two different) programs confirmed this list. Looking carefully at the results leads us to two natural questions ; we could answer one of these positively, but not the other one.

Theorem 4. $\forall N \geq 7$, $Q(N, 3, \leq) = Q_+(N, 3, \leq) = N + 1 - \left\lfloor \frac{N+1}{4} \right\rfloor$.

Question 1. *Is it true that for any positive integer $p \geq 4$ there exists a smallest integer $N(p) \geq 2p+1$ such that for any integer $N \geq N(p)$ we have:*
$Q(N, p, \leq) = Q_-(N, p, \leq) \iff N \equiv r\ [p+1]$ *for* $\frac{p+1}{2} < r \leq p-1$ *?*

Using the program Main.hs, available from Wildon's website[1], generating all couples (N, p) such that $Q(N, p, \leq) = Q_-(N, p, \leq)$ for $N \leq 51$, we could verify that there is no contradiction to a positive answer to Question 1. Furthermore we could prove that the property is verified for any even $p \leq 12$ with $N(4) = 9$, $N(6) = 19$, $N(8) = 24$, $N(10) = 29$ and $N(12) = 34$. Notice that, by the proof of Theorem 3, show that for any p there exists some $N \equiv \lfloor \frac{p+1}{2} \rfloor + 1$ such that $Q(N, p, \leq) = Q_-(N, p, \leq)$ would be enough to answer positively Question 1 in case p is even. For the case where p is odd one should furthermore show that $Q(N, p, \leq) = Q_+(N, p, \leq)$ for every $N \equiv \frac{p+1}{2}\ [p+1]$.

[1] www.ma.rhul.ac.uk/ uvah099/Programs/MajorityGame/Main.hs

3 The $(N, p, =)$-identification problem

$Q(N, p, =)$ happens to be more difficult to estimate than $Q(N, p, \leq)$. We could obtain one upper bound.

Theorem 5. *Let N and p be integers such that $1 \leq p < \frac{N}{2}$, and let m be the largest integer such that $2^m \leq \min(N - 2p, 2p)$, we have :*

$$Q(N, p, =) \leq N + 1 - m - \left\lfloor \frac{N + 2 - 2^m}{p + 1} \right\rfloor .$$

For $p = 1$ or 2 we were able to compute the exact value of $Q(N, p, =)$.

Theorem 6. $\forall N \geq 3$, $Q(N, 1, =) = N - \left\lfloor \frac{N}{2} \right\rfloor$.

Theorem 7. $\forall N \geq 6$, $Q(N, 2, =) = N - \left\lfloor \frac{N+1}{3} \right\rfloor$.

Theorems 6 and 7 show that, for $p = 1$ or 2, the upper bound $Q_+(N, p, =)$ of $Q(N, p, =)$ given by Theorem 5, is in fact the right value of $Q(N, p, =)$. This is however not true for any value of p as shown by our programs. For $N \leq 30$ and $p < \frac{N}{2}$ it happens several times that $Q(N, p, =) = Q_+(N, p, =) - 1$, once that $Q(N, p, =) = Q_+(N, p, =) - 2$ (for $(N, p) = (30, 4)$) and for all other couples (N, p) the equality $Q(N, p, =) = Q_+(N, p, =)$ holds.

For $p \geq 3$, it seems that the upper bound $Q_+(N, p, =)$ works well for "small" values of N's, however we defined a family of methods that we call "Tower methods" that are more efficient in the case of "very big" N's. However one may also wonder if there are other methods that could give better bounds.

References

[1] M. Aigner. Variants of the Majority Problem. Discrete Applied Mathematics 137 Issue 1 (2004). 3–25.

[2] M. Wildon, Searching for knights and spies: A majority/minority game. Discrete Math. 339 (2016), 2754–2766.