Fibonacci numbers of graphs composition on some circulant graphs[1]

Loiret Alejandría Dosal-Trujillo — Instituto de Matemáticas, UNAM
Hortensia Galeana-Sánchez — Instituto de Matemáticas, UNAM

Abstract

The Fibonacci number of a graph G, is the total number of independent vertex sets of G and it is denoted by $\mathscr{F}(G)$. This concept was first defined by Prodinger and Tichy in 1982. A closely related concept is the independence polynomial of a graph, introduced in 1983 by Gutman and Harary. The problem of finding the Fibonacci number of a graph is an NP-complete problem. In 2017 we proved that for every two graphs G and H, the Fibonacci number of $G[H]$, the composition of G with H, can be obtained in terms of the independence polynomial of G and the Fibonacci number of H. A graph G is said to be vertex-transitive if the automorphisms group of G acts transitively in its set of vertices. In this talk we will show how to calculate the Fibonacci number of $C_{n[r]}[H]$, the composition of a circulant graph of order n and consecutive jumps $(1, 2, \ldots, r)$ with a graph H, using the fact that $C_{n[r]}$ is a vertex-transitive graph.

Let $G = (V, E)$ a graph. A subset S of V is said to be independent if for every two vertices $u, v \in S$ there is no edge between them. For a vertex v, $G-v$ and $G-N[v]$ are the graphs obtained from G by deleting the vertex v and the set $N[v]$, respectively; where $N[v]$ denotes the set that contains v and all vertices of G adjacent to v. $N[v]$ is called the closed neighborhood of v. Let I_G be the set of all independent vertex subsets of G, and v any fixed vertex of G. Consider the following sets, I_{G_v} the set of all elements of I_G containing v and let $I_{G_{-v}}$ its complement. Since $\{I_{G_v}, I_{G_{-v}}\}$ is a partition of I_G, then the Fibonacci number of G can be obtained by counting all independent sets containing or not the vertex v. This usual counting method to determine the Fibonacci number of a graph was provided for Tichy and Prodinger in [5], they proved that the number of independent vertex sets of P_n, the path of order n is precisely a Fibonacci number, and for C_n the cycle of order n is the n-Lucas number[2].

Lemma 1 ([5]). *Let G be a graph and v a vertex of G. Then*

$$\mathscr{F}(G) = \mathscr{F}(G - v) + \mathscr{F}(G - N[v]).$$

Given a graph G and a family of graphs $\alpha = (\alpha_v)_{v \in V(G)}$ without vertices in common, the Zykov sum $\sigma(G, \alpha)$ is the graph with vertex set $\cup_{v \in V(G)} V(\alpha_v)$, and edge set $\cup_{v \in V(G)} E(\alpha_v) \cup \{xy : x \in V(\alpha_u), y \in V(\alpha_v)$ and $uv \in E(G)\}$. The composition of two disjoint graphs G with H, denoted by $G[H]$ is the graph $\sigma(G, \alpha)$, where α_v is isomorphic to the graph H, for every $v \in V(G)$.

In [2] we proved a generalization of Lemma 1:

Theorem 1 ([2]). *Let G and H be two graphs and $v \in V(G)$,*

$$\mathscr{F}(G[H]) = \mathscr{F}((G - v)[H]) + (\mathscr{F}(H) - 1) \cdot \mathscr{F}((G - N[v])[H]).$$

The independence polynomial of a graph G of order n, is defined as follows: $P_x(G) = a_0 + a_1 x + \cdots + a_n x^n$, where a_i is the number of independent vertex-sets of G, with cardinality i, for

[1]This research was supported by grant UNAM-DGAPA-PAPIIT IN104717.

[2]For a description of Fibonacci and Lucas numbers, and some nice intrinsic relationships between them, see [4]).

all $i \in \{0, \ldots, n\}$. Observe that $a_0 = 1$ and $a_1 = n$ for every graph G of order n and $a_i = 0$ for all $i > \alpha(G)$, where $\alpha(G)$ denotes the maximum cardinality of an independent vertex set of G. We call $\alpha(G)$ the independence number of G. And it is clear that $\mathscr{F}(G) = \sum_{i=1}^{n} a_i$, where a_i are the coefficients of $P_x(G)$. Gutman and Harary provides the following result of independence polynomial of graphs in [3].

Lemma 2 ([3]). *For every graph G and an arbitrary fixed vertex $v \in V(G)$,*

$$P_x(G) = P_x(G - u) + xP_x(G - N[v]).$$

In [2] we proved the following results on independence polynomial of graphs composition:

Theorem 2 ([2]). *Let G and H be two graphs, $G[H]$ its composition and v a vertex of G, then*

$$P_x(G[H]) = P_x((G - v)[H]) + (P_x(H) - 1)P_x((G - N[v])[H]).$$

Given a graph G, and $P_x(G)$ its independence polynomial, $P_x(G)[y]$ denotes the independence polynomial of G evaluated in y, with $y \in \mathbb{Z}$.

Theorem 3 ([2]). *If G and H are two vertex disjoint graphs, then*

$$\mathscr{F}(G[H]) = P_x(G)[\mathscr{F}(H) - 1].$$

A circulant graph $C_{n(m_1,m_2,\ldots,m_r)}$ is a graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_{i+m_j \pmod n} : i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, r\}\}$, where $r \in \mathbb{Z}^+$. The values m_j are the jump sizes. $C_{n[r]}$ denotes the circulant graphs $C_{n(1,2,\ldots,r)}$. In [1] we proved that for every $r \in \mathbb{Z}^+$ and for all $n \geq r+1$, the Fibonacci numbers of $C_{n[r]}$ and for several subgraphs of this family are characterized by some recursive sequences which generalize the Fibonacci and Lucas sequences.

Let G be a graph and $v \in V(G)$, we define the k-Fibonacci number of v, denoted by $f_k(v)$, as the number of independent sets of G with cardinality $k \geq 1$, in wich v belongs. A graph G is vertex-transitive if for every two vertices u and v there is an automormorphism ϕ of G, such that $\phi(u) = v$.

Main results:

Theorem 4 (Dosal-Trujillo and Galeana-Sánchez, 2018). *Let G be a vertex-transitive graph of order n with independence number $\alpha(G)$, and $f_k(v)$ the k-Fibonacci number of a vertex v,*

$$P_x(G) = 1 + \sum_{k=1}^{\alpha(G)} \frac{n}{k} f_k(v) x^k.$$

Theorem 5 (Dosal-Trujillo and Galeana-Sánchez, 2018). *Let $r \in \mathbb{Z}^+$ and α_r the independence number of $C_{n[r]}$. For all $n \geq 2r + 2$ and for every graph H,*

$$\mathscr{F}(C_{n[r]}[H]) = 1 + \sum_{k=1}^{\alpha_r} \frac{n}{k} \binom{n - (rk + 1)}{k - 1} (\mathscr{F}(H) - 1)^k.$$

References

[1] L. A. Dosal-Trujillo and H. Galeana-Sánchez. The Fibonacci numbers of certain subgraphs of Circulant graphs. AKCE International Journal of Graphs and Combinatorics *12* (2015) 94–103.

[2] L. A. Dosal-Trujillo and H. Galeana-Sánchez. On the Fibonacci numbers of the composition of graphs. Submmited.

[3] I. Gutman and F. Harary. Generalizations of the matching polynomial. Utilitas Mathematica **24** (1983) 97–106.

[4] T. Koshy. Elementary Number Theory with Applications. Academic Press, Burlington, 2007.

[5] H. Prodinger and R.F. Tichy. Fibonacci numbers of graphs. The Fibonacci Quarterly **20** (1982), 16–21.