Strong duality and mutual transferability for mixed domination on strongly chordal graphs and cactus graphs

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Abstract

Let $G = (V, E)$ be a graph and V be the disjoint union of F, B, and R. A mixed dominating set of G is a subset $D \subseteq V$ such that $R \subseteq D$ and each vertex in $B - D$ is adjacent to some vertex in D. A mixed 2-stable set of G is a subset $S \subseteq B$ such that $S \cap N[R] = \emptyset$ and every two distinct vertices x and y in S have distance $d(x, y) > 2$. We prove that if G is strongly chordal, then $\alpha_{\mathrm{m},2}(G) = \gamma_{\mathrm{m}}(G) - |R|$, where $\gamma_{\mathrm{m}}(G)$ is the minimum cardinality of a mixed dominating set of G and $\alpha_{\mathrm{m},2}(G)$ is the maximum cardinality of a mixed 2-stable set of G. Let D_1 and D_2 be any mixed dominating sets of G with $|D_1| = |D_2|$. We also prove that if G is connected strongly chordal, then D_1 and D_2 are mutually transferable; and if G is a cactus graph, then adding one extra element can ensure that D_1 and D_2 are mutually transferable.

Domination and its variations in graphs have been widely studied in the literature and they have numerous applications in computer networks. Especially, they can be used in server allocation to ensure the Quality of Service (QoS) of a network.

Let $G = (V, E)$ be a graph. The *neighborhood* $N(v)$ of a vertex v is the set of all vertices adjacent to v; the *closed neighborhood* $N[v]$ of v is $N[v] = N(v) \cup \{v\}$. A vertex v is said to *dominate* all vertices in $N[v]$. A subset $D \subseteq V$ is a *dominating set* of G if every vertex in $V - D$ is adjacent to at least one vertex in D. A subset $S \subseteq V$ is a *2-stable set* of G if every two distinct vertices x and y in S have $d(x, y) > 2$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The *2-stability number* $\alpha_2(G)$ of G is the maximum cardinality of a 2-stable set of G. It is easy to see that any graph G has (Weak Duality Inequality for Domination): $\alpha_2(G) \leq \gamma(G)$. Chang [2] used an algorithmic approach to show that any tree T has (Strong Duality Equality for Domination): $\alpha_2(T) = \gamma(T)$. Such a primal-dual approach was used by Farber [8] and Kolen [11] for weighted domination on strongly chordal graphs.

Let D_1 and D_2 be two dominating sets of G. We say that D_1 is *single-step transferable* to D_2, denoted as $D_1 \to D_2$, if there exist $u \in D_1$ and $v \in D_2$ such that $uv \in E$ and $D_1 - \{u\} = D_2 - \{v\}$. We write $D_1 \overset{*}{\to} D_2$ if D_1 can be transferred to D_2 through a sequence of single-step transfers (also called a sequence of *safe moves* in [9]). See Fig. 1. By definition, $D_1 \to D_2$ implies that $D_2 \to D_1$ and $|D_1| = |D_2|$; also, $D_1 \overset{*}{\to} D_2$ implies that $D_2 \overset{*}{\to} D_1$ and $|D_1| = |D_2|$. For technical reasons, [9] assumes that a dominating set is a dominating multiset; in this manuscript, we make the same assumption. Reference [9] studies the following server allocation problem:

> Given two dominating sets D_1 and D_2 of a graph G, can we transfer D_1 to D_2 through a sequence of safe moves?

The main results of [9] are as follows.

- For any tree T with n vertices, if D_1 and D_2 are two dominating sets of T and $|D_1| = |D_2| \geq \lfloor \frac{n}{2} \rfloor$, then $D_1 \overset{*}{\to} D_2$.
- For any Hamiltonian graph G with n vertices, if D_1 and D_2 are two dominating sets of G and $|D_1| = |D_2| \geq \lceil \frac{n+1}{3} \rceil$, then $D_1 \overset{*}{\to} D_2$.

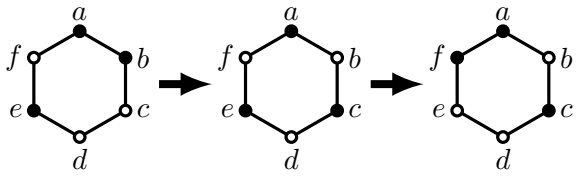


Figure 1: Let $D_1 = \{a, b, e\}$, $D_2 = \{a, c, f\}$, and $D_3 = \{a, c, e\}$. Then, $D_1 \to D_3$, $D_3 \to D_2$, and $D_1 \overset{*}{\to} D_2$.

The purpose of this manuscript is to study mixed domination for strongly chordal graphs and cactus graphs. Let $G = (V, E)$ be a graph and V be the disjoint union of F, B, and R, where F means "*Free*", consisting of free vertices, B means "*Bound*", consisting of bound vertices, and R means "*Required*", consisting of required vertices. A *mixed dominating set* D of G is a subset $D \subseteq V$ such that $R \subseteq D$ and each vertex in $B - D$ is adjacent to some vertex in D; free vertices need not be dominated by D but may be included in D to dominate bound vertices. (See [5, 10].) Notice that ordinary domination is the same as mixed domination when $B = V$, $F = R = \emptyset$. Let mixed 2-stable set of G, $\gamma_{\rm m}(G)$, and $\alpha_{\rm m,2}(G)$ be defined as in the abstract. By replacing "dominating set" with "mixed dominating set" in the definitions of $D_1 \to D_2$ and $D_1 \overset{*}{\to} D_2$, mutual transferability among mixed dominating sets D_1 and D_2 of G can be defined and we omit the details.

Before going further, we define the notion $\cup\{1 \cdot \text{extra}\}$. We actually define the notion $D \cup \{\ell \cdot \text{extra}\}$. Intuitively, when a mixed dominating set of G contains more than $\gamma_{\rm m}(G)$ elements, it may contain elements whose deletion will not affect the mixed domination and we call these elements *extra elements*. More precisely, suppose D is already a mixed dominating set of G and $\{s_1, \ldots, s_\ell\}$ is a subset of $V(G)$. Then, $D \cup \{s_1, \ldots, s_\ell\}$ is a mixed dominating set of G and $s_1, \ldots, s_\ell$ are extra elements. When the names of $s_1, \ldots, s_\ell$ are unimportant, we simply write $D \cup \{s_1, \ldots, s_\ell\}$ as $D \cup \{\ell \cdot \text{extra}\}$. By setting $\ell = 1$ into $D \cup \{\ell \cdot \text{extra}\}$, the notion $\cup\{1 \cdot \text{extra}\}$ is now clear. Notice that when $D_1 \cup \{1 \cdot \text{extra}\} \overset{*}{\to} D_2 \cup \{1 \cdot \text{extra}\}$, a sequence of single-step transfers from D_1 to D_2 can contain a mixed dominating set of G that has no extra element; D_3 in Fig. 1 is an example.

A *strong elimination ordering* of a graph $G = (V, E)$ is an ordering $v_1, v_2, \ldots, v_n$ of V such that for each i, j, and k, if $i \le j \le k$ and $v_j, v_k \in N(v_i)$, then $N[v_j] \cap \{v_i, v_{i+1}, \ldots, v_n\} \subseteq N[v_k] \cap \{v_i, v_{i+1}, \ldots, v_n\}$. A graph is *strongly chordal* if it admits a strong elimination ordering. The class of strongly chordal graphs was introduced by Farber in [6, 7, 8]. Chang and Nemhauser [1, 3] independently discovered this class of graphs. It is easy to see the **Weak Duality Inequality for Mixed Domination:** $\alpha_{\rm m,2}(G) \le \gamma_{\rm m}(G) - |R|$ for any graph G. The inequality can be strict, as shown by the 5-cycle C_5 (with B being the set of all vertices and $F = R = \emptyset$) that $\alpha_{\rm m,2}(C_5) = 1 < 2 = \gamma_{\rm m}(C_5) - |R|$.

Main results of this manuscript are stated in Theorem 1, Theorem 2, and Theorem 3.

Theorem 1. (Strong Duality Equality for Mixed Domination): *Any strongly chordal graph G has $\alpha_{\rm m,2}(G) = \gamma_{\rm m}(G) - |R|$.*

The proof of Theorem 1 requires Algorithm 1, which finds both a minimum mixed dominating set D^* and a maximum mixed 2-stable set S^* of a strongly chordal graph G.

Sketch of the Proof of Theorem 1. It is clear that $D^* = R \cup \{r_1, r_2, \ldots, r_k\}$, $S^* = \{b_1, b_2, \ldots, b_k\}$, and D^* is a mixed dominating set of G. Notice that it can be proven that $S^* \cap N[R] = \emptyset$ and every two distinct b_i and b_j in S^* have $d(b_i, b_j) > 2$. Therefore S^* is a mixed 2-stable set of G.

Algorithm 1

Input: A strongly chordal graph $G = (V, E)$ with a strong elimination ordering $v_1, v_2, \ldots, v_n$ and a partition F, B, R of V.

Output: A minimum mixed dominating set D^* and a maximum mixed 2-stable set S^* of G.

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1: D* ← R;  S* ← ∅;  relabel all the bound vertices in N(R) as free;
2: k ← 0;
3: for i = 1 to n do
4:     if (v_i is bound) then
5:         k ← k + 1;  b_k ← v_i;  r_k ← the largest-indexed vertex v_{i*} in N[v_i];
6:         relabel v_{i*} as required;  relabel all the bound vertices in N(v_{i*}) as free;
7:         D* ← D* ∪ {v_{i*}};  S* ← S* ∪ {v_i};
8:     end if
9: end for
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Thus $|S^*| \leq \alpha_{\mathrm{m},2}(G)$ and $\gamma_{\mathrm{m}}(G) \leq |D^*|$. By the weak duality inequality for mixed domination, $\alpha_{\mathrm{m},2}(G) \leq \gamma_{\mathrm{m}}(G) - |R|$. Since $R \cap \{r_1, r_2, \ldots, r_k\} = \emptyset$, we have $D^* - R = \{r_1, r_2, \ldots, r_k\}$ and $|D^*| - |R| = |D^* - R|$. Thus $k = |\{b_1, b_2, \ldots, b_k\}| = |S^*| \leq \alpha_{\mathrm{m},2}(G) \leq \gamma_{\mathrm{m}}(G) - |R| \leq |D^*| - |R| = |D^* - R| = |\{r_1, r_2, \ldots, r_k\}| = k$. The three inequalities must be equalities. Hence D^* is a minimum mixed dominating set and S^* is a maximum mixed 2-stable set of G. So $\alpha_{\mathrm{m},2}(G) = \gamma_{\mathrm{m}}(G) - |R|$. □

The second and the third main results of this manuscript are:

Theorem 2. *For any connected strongly chordal graph G, if D_1 and D_2 are two mixed dominating sets of G with $|D_1| = |D_2|$, then $D_1 \overset{*}{\to} D_2$.*

Theorem 3. *For any cactus graph G, if D_1 and D_2 are two mixed dominating sets of G with $|D_1| = |D_2|$, then $D_1 \cup \{1 \cdot extra\} \overset{*}{\to} D_2 \cup \{1 \cdot extra\}$.*

By letting $B = V(G)$ and $F = R = \emptyset$, we have:

Corollary 1. *If G is a connected strongly chordal graph in which D_1 and D_2 are two dominating sets with $|D_1| = |D_2|$, then $D_1 \overset{*}{\to} D_2$.*

Corollary 2. *If G is a cactus graph in which D_1 and D_2 are two dominating sets with $|D_1| = |D_2|$, then $D_1 \cup \{1 \cdot extra\} \overset{*}{\to} D_2 \cup \{1 \cdot extra\}$.*

It is well known that strongly chordal graphs is a subclass of chordal graphs. On the other hand, trees, block graphs, interval graphs, and directed path graphs are subclasses of strongly chordal graphs. Containment relations between these graphs are:

$$\begin{array}{rcccccc} \text{tree} & \subset & \text{block graph} & \subset & \text{strongly chordal graph} & \subset & \text{chordal graph} \\ \text{interval graph} & \subset & \text{directed path graph} & \subset & \text{strongly chordal graph} & \subset & \text{chordal graph} \end{array}$$

It follows that our second main result applies to trees, block graphs, interval graphs, and directed path graphs. Thus our second main result improves the first main result of [9], which only applies to trees and dominating sets of cardinality $\geq \lfloor \frac{n}{2} \rfloor$. Every cycle C_{3r+6}, $r \geq 0$, is a cactus graph. However, no two distinct minimum dominating sets of C_{3r+6} are mutually transferable; therefore at least one extra element has to be added. Our third main result (which adds only one extra element) is therefore the best possible.

Up to now, finding a minimum mixed dominating set is regarded as a labeling method for deriving a minimum dominating set. The final goal is a minimum dominating set, not a minimum mixed dominating set. However, as can be seen in the following scenario, a minimum mixed dominating set is more flexible in modeling a real-world problem. Some of the cities in Taiwan are the most important (for example, Taipei and Hsinchu); we need to allocate a server for each of them and therefore mark them as required. Some of the cities in Taiwan are important but not the most important; we don't need to allocate a server for each of them but we need to ensure that each of them has a server within its closed neighborhood and therefore we mark them as bound. Some of the cities in Taiwan are not so important and we don't even have to make each of them have a server within its closed neighborhood; we therefore mark them as free.

For any connected graph G, we may try to determine the least number $\ell(G)$ of extra elements required to guarantee mutually transferability. That is, if D_1 and D_2 are two mixed dominating sets of G with $|D_1| = |D_2|$, then $D_1 \cup \{\ell \cdot \text{extra}\} \xrightarrow{*} D_2 \cup \{\ell \cdot \text{extra}\}$ for $\ell \geq \ell(G)$. For example, $\ell(G) = 0$ when G is connected strongly chordal graph and $\ell(G) = 1$ when G is a cactus graph. We conjecture that any connected graph G has $\ell(G) \leq 1$. In fact, for cycles C_n, if all vertices are labeled as bound (or equivalently, ordinary domination), then $\ell(C_n) = 1$ if $n = 3r + 6$ and $\ell(C_n) = 0$ otherwise. (Note that a preliminary version of this manuscript was in [4].)

Acknowledgement. The authors would like to thank Prof. Gerard Jennhwa Chang for his valuable comments. This research was partially supported by the Ministry of Science and Technology of Taiwan under grants MOST-106-2115-M-009-005 and MOST-106-2115-M-009-007.

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