The swap Markov chain on bipartite degree sequences with bounded maximum degree is rapidly mixing

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Abstract

In the last two decades the **swap** (switch) Markov chain approach was heavily used to sample realizations of graphic degree sequences. It is conjectured that this Markov chain is rapidly mixing on all sequences, however there are particulary few proven cases. Here we report two new results:

(1) Let d be a bipartite graphical degree sequence, and let $\Delta = \max d$. Let M be half of the sum of the degrees. If $2 \leq \Delta \leq \frac{1}{\sqrt{2}}\sqrt{M}$, then the swap Markov Chain on the realizations of d is rapidly mixing.

(2) Let $\vec{d}$ be a directed degree sequence. Let M be half of the sum of the degrees. If

$$\max\{\max d_{out}, \max d_{in}\} < \sqrt{M/2 - 2},$$

then the swap Markov chain on this directed degree sequence is rapidly mixing.

These results are germane to the recent results of Greenhill and Sfragara about rapidly mixing MCMC processes on simple and directed degree sequences.

1 Introduction and history

An important problem in network science is to algorithmically construct typical instances of networks with predefined properties, often expressed as graph measures. In particular, special attention has been devoted to sampling simple graphs with a given degree sequence. In 1997 Kannan, Tetali, and Vempala ([8]) proposed the use of the so-called **swap Markov chain** approach, which had already been used in statistics.

The **swap operation** (also known as switch operation) exchanges two disjoint edges ac and bd in the realization G with ad and bc if the resulting configuration G' is again a simple graph (we denote the operation by $ac, bd \Rightarrow ad, bc$). It is a well-known fact that the set of all possible realizations of a **graphic** degree sequence is connected under this operation. (See Havel [7].) Similar operations can be considered for bipartite graphs with similar properties. Sometimes not every edge exchange is potentially possible, for example, on a bipartite graph we must ensure that vertices a and d belong to different vertex classes.

In case of directed graphs, to achieve full analogy the introduction of the so-called **triple swap** operation is required. Indeed, imagine that our graph $\vec{H}_1$ is a directed triangle $\overrightarrow{C_3}$ while $\vec{H}_2$ is the oppositely directed $\overleftarrow{C_3}$. Both graphs have the same degree bi-sequence $\mathbf{D} = ((1,1,1);(1,1,1))$. (The first vector is the list of the in-degrees, while the second one is the list of the out-degrees.) It is clear that in order to transform $\vec{H}_1$ into $\vec{H}_2$ we should exchange three edges and three non-edges (For details see for example [2].)

The simple directed graph $\vec{G}$ can be represented by a bipartite graph (see [4]): let $B(\vec{G}) = (U, V; E)$ be a bipartite graph where each class consists of one copy of every vertex x of the vertex set $X(\vec{G})$. The edges adjacent to a vertex u_x in class U represent the out-edges from x, while the edges adjacent to a vertex v_x in class V represent the in-edges to x (so a directed edge $\overrightarrow{xy}$

corresponds the edge $u_x v_y$). Since there are no loops in a simple directed graph, there cannot be a (u_x, v_x) edge in its bipartite representation — these vertex pairs are **forbidden** and form a forbidden 1-factor in the bipartite graph $B(\vec{G})$.

In the bipartite representations of the directed graphs $\vec{H}_1$ and $\vec{H}_2$ no edge pair can be transformed by a swap operation. In the bipartite representations the edges and non-edges of an alternating C_6 need to be exchanged (also called a triple swap) to transform $B(\vec{H}_1)$ into $B(\vec{H}_2)$. (For details see [2].)

The swap Markov chains corresponding to the most common graph models are irreducible, aperiodic, reversible (obey detailed balance), have symmetric transition matrices, and thus have uniform global stationary distributions.

In their paper [8], Kannan, Tetali and Vempala conjectured that all these Markov chains are rapidly mixing. The first rigorous proof in this topic is due to Cooper, Dyer and Greenhill about regular simple graphs ([1]). The following known degree sequence classes accommodate rapidly mixing swap Markov chains:

1. simple regular degree sequences (see [6]);
2. regular directed degree sequences (see [6]);
3. half regular bipartite degree sequences (it means that in one class the degrees are the same, while in the other class the only restrictions are those imposed by graphicality) (see [2]);
4. almost half regular bipartite degree sequences (here on one side we have $|d(v_1) - d(v_2)| \leq 1$) (see [2]);
5. almost half regular directed degree sequences (see [2]).

Recently the following results were found and published (see [3]): Let $\mathbf{d}$ be a bipartite degree sequence on the underlying set $U \uplus V$.

Theorem 1. *Let $0 < c_1 \leq c_2 < |U| = n$ and $0 < d_1 \leq d_2 < |V| = m$ be integer parameters and assume $\mathbf{d}$ satisfies the following properties:*

$$\begin{aligned} c_1 \leq d(v) \leq c_2, \qquad & \forall v \in V \\ d_1 \leq d(u) \leq d_2, \qquad & \forall u \in U. \end{aligned} \tag{1}$$

Furthermore, assume that

$$(c_2 - c_1 - 1) \cdot (d_2 - d_1 - 1) \leq \max\{c_1(m - d_2), d_1(n - c_2)\} \tag{2}$$

holds. Then the swap Markov chain on the realizations of this bipartite degree sequence is rapidly mixing.

The analogous result for directed degree sequences is the following: Let $\vec{\mathbf{d}}$ a directed degree sequence on the n element vertex set V.

Theorem 2. *Let $0 < c_1 \leq c_2 < n$ and $0 < d_1 \leq d_2 < n$ be integer parameters and assume that graphic degree be-sequence $\vec{\mathbf{d}}$ satisfies the following properties:*

$$\begin{aligned} c_1 \leq d_{out}(v) \leq c_2, \qquad & \forall v \in V, \\ d_1 \leq d_{in}(v) \leq d_2, \qquad & \forall v \in V. \end{aligned} \tag{3}$$

Furthermore, assume that

$$(c_2 - c_1) \cdot (d_2 - d_1) \leq 2 + \max\Big\{c_1(n - d_2 - 1) + d_1 + c_2, \quad d_1(n - c_2 - 1) + c_1 + d_2\Big\} - n \qquad (4)$$

holds. Then the swap Markov chain using double and triple swap operations is rapidly mixing on the realizations of this directed degree sequence.

In the same paper it was also shown that on bipartite and directed degree sequences generated under the Erdős – Rényi random model (with not extremely low or high edges probabilities), the swap Markov chains are rapidly mixing.

2 New results

In [6] Greenhill and Sfragara developed the following nice results:

Theorem 3 (Greenhill, Sfragara (2017))**.** *(i) Let* $\mathbf{d}$ *be a simple graphic degree sequence, satisfying* $3 \leq d_{\max} \leq \frac{1}{3}\sqrt{M}$*, where* M *is the sum of the degrees. Then the swap Markov chain is rapidly mixing.*
(ii) Let $\mathbf{d}$ *is a graphic directed degree sequence with* $2 \leq d_{\max} \leq \frac{1}{4}\sqrt{M}$*, where* M *is the sum of the degrees, and where the set of all realizations under study is irreducible under the double swap operation. Then the swap Markov chain is rapidly mixing.*

Due to the irreducibility condition, (ii) does not apply to all degree sequences numerically satisfying the inequality. One example for such a degree sequence is the already mentioned $\mathbf{D} = ((1,1,1);(1,1,1))$. However these results cover a wide range of degree sequences, including power-law graphs with parameter $\gamma > 5/2$ and sufficiently many edges (see [6]).

In what follows we announce two new analogous results.

Theorem 4. *Let* $\mathbf{d}$ *be a bipartite degree sequence, let* M *be half of the sum of the degrees in* $\mathbf{d}$*, and let* $\Delta = \max d$*. If*

$$2 \leq \Delta \leq \frac{1}{\sqrt{2}}\sqrt{M}, \qquad (5)$$

then the corresponding Markov chain is rapidly mixing on the realizations of $\mathbf{d}$*.*

Theorem 5. *Let* $\vec{d}$ *be a directed degree sequence. Let* M *be half of the sum of the degrees, and let* $\Delta = \max\{\max d_{out}, \max d_{in}\}$*. If*

$$\Delta < \frac{1}{\sqrt{2}}\sqrt{M-4}, \qquad (6)$$

then the corresponding Markov chain is rapidly mixing on the realizations of $\vec{d}$*.*

This theorem, unlike Theorem 3, applies to all directed degree sequences satisfying the numerical conditions. This is a consequence of the fact that our Markov chain uses both double and triple swaps. The small discrepancy between inequalities (5) and (6) is due to the forbidden 1-factor which the realizations must respect.

The proof of these results are rather similar to the proofs of Theorem 1 and 2. We will illustrate it with a bird's-eye view of the more complicated proof of Theorem 5. The Markov chain is constructed the same way as in [3], including the transition matrix ([3, condition (2.2)]).

For any pair of realizations G_1 and G_2 we have to construct the multicommodity flow of Sinclair ([9]), satisfying the conditions of [3, Theorem 3.1]. To that end we consider the symmetric difference of realizations G_1 and G_2. We decompose it into ordered alternating cycles in all possible ways. (The word alternating indicates that the cycles contain edges from G_1 and G_2 in turns.

In the next step we fix a particular cycle decomposition and determine a unique canonical path from G_1 to G_2 corresponding to this decomposition. To construct a part of the path, we take an alternating cycle of the decomposition and we design an eligible swap sequence to process this cycle (i.e., to exchange its G_1 edges to G_2 edges). This process is identical to that of [3, Section 4].

The next step is to analyze the constructed swap sequences and prove that they satisfy the conditions of Theorem 3.1 in [3]. The fundamental point of that is a counting lemma (this terminology coined by Catherine Greenhill) which proves that the edges representing the swap operations between realizations are not oversaturated by the canonical paths. As Greenhill pointed out in [5], this is the centerpiece of all known methods to prove the rapidly mixing nature of a swap Markov chain.

This particular part of the proof of Theorem 5 is very similar to the proof of [3, Lemma 6.1]. We can use the same auxiliary structures and similar reasoning, but the actual calculations are of course different. This finishes the very high level sketch of the proof. □

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