# The sandwich conjecture of random regular graphs and more

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**Abstract**

The sandwich conjecture formulated in [Kim, Vu, Advances in Math., 2004] states that if $\min\{d, n-d\} \geq \log n$, then the random $d$-regular graph on $n$ vertices can asymptotically almost surely be "sandwiched" between $G(n, p_1)$ and $G(n, p_2)$ where probabilities $p_1$ and $p_2$ are both $(1 + o(1))d/n$. They proved this conjecture for the range $\log n \leq d \leq n^{1/3-o(1)}$ with a defect in one side of sandwiching: a few edges from each vertex should be deleted from the random regular graph to guarantee the containment. Recently, their result (one-sided containment) was improved by Dudek, Frieze, Ruciński and Šileikis to $d = o(n)$.

We prove the sandwich conjecture (with perfect containments on both sides) for all values of $d$ such that $\min\{d, n-d\} \gg n/\log n$. In this talk we also discuss extensions to random subgraphs of a given graph and, in particular, random regular bipartite graphs.

## 1 Introduction

Random graphs are one of the most important notions in modern graph theory. Besides the rich theory in its own field, it has many connections and applications in the general area of combinatorics. Many existence combinatorial results are shown by using or modifying random graphs. Today random graphs are also widely used in computer science, engineering, physics and other branches of sciences.

There are many random graph models. The most classical models $\mathcal{G}(n, p)$ and $\mathcal{G}(n, m)$ are introduced by Erdős and Renyi more than half a century ago. The binomial model $\mathcal{G}(n, p)$ retains each potential edge in the complete graph $K_n$ independently with probability $p$. The $\mathcal{G}(n, m)$ is simply $\mathcal{G}(n, p)$ by restricting to having exactly $m$ edges in total. In other words, $\mathcal{G}(n, m)$ is the random graphs on $n$ vertices and $m$ edges with the uniform distribution. These two models are the best studied and understood. The independence between the occurrence of edges makes $\mathcal{G}(n, p)$ a relative easier model, compared with many others, for analysing its properties, and for analysing algorithms on $\mathcal{G}(n, p)$. Some algorithms depend on the degrees of vertices, and unavoidably the algorithms need to "expose" the degrees of the vertices as the algorithms proceed. For instance, the peeling algorithm for obtaining the $k$-core of the graph repeated deletes a vertex whose degree is below $k$. An important property of $\mathcal{G}(n, p)$ and $\mathcal{G}(n, m)$ is that, by conditioning on the degree sequence of $\mathcal{G}(n, p)$ or $\mathcal{G}(n, m)$ being $\mathbf{d} = (d_1, \dots, d_n)$, the resulting random graph is exactly $\mathcal{G}(n, \mathbf{d})$, the random graph given degree sequence $\mathbf{d}$, with the uniform distribution. This places $\mathcal{G}(n, \mathbf{d})$ among the most important random graph models in the study of random graphs. If $\mathbf{d}$ is a constant sequence, then we write $\mathcal{G}(n, d)$ for the random $d$-regular graph. Unlike $\mathcal{G}(n, p)$, probabilities of events in $\mathcal{G}(n, \mathbf{d})$ such as two vertices $u$ and $v$ being adjacent are highly non-trivial. The most common way to prove properties of $\mathcal{G}(n, \mathbf{d})$ is via a translation result from the configuration model [1] to the random graph model. Probabilities of events are usually easier to compute or estimate in the configuration model than in $\mathcal{G}(n, \mathbf{d})$. However, such a translation result can only be applied when degrees specified by $\mathbf{d}$ are small. When the average degree grows with $n$, translation becomes hard or fails.

As a consequence, many questions that feel like deserving a confirmative answer are indeed open for $\mathcal{G}(n, \mathbf{d})$. For instance, is $\mathcal{G}(n, \mathbf{d})$ Hamiltonian? What is the chromatic number of $\mathcal{G}(n, \mathbf{d})$? What is the connectivity of $\mathcal{G}(n, \mathbf{d})$? Using highly non-trivial switching arguments and enumeration results of $d$-regular graphs, these questions were answered [2, 9] for $\mathcal{G}(n, d)$. Using similar techniques it may be possible to work out the answers for more general degree sequences. However, it will be desirable to have simpler approaches.

This is the motivation of the sandwich conjecture, proposed by Kim and Vu [8] in 2004. They conjectured that for every $d \gg \log n$, $\mathcal{G}(n, d)$ can be sandwiched by two binomial random graphs $\mathcal{G}(n, p_1)$ and $\mathcal{G}(n, p_2)$, one with slightly smaller average degree, and the other with slightly greater average degree than $d$. The formal statement is as follows.

**Conjecture 1** (Sandwich Conjecture [8])**.** *For $d \gg \log n$, there are $p_1 = (1 - o(1))d/n$ and $p_2 = (1 + o(1))d/n$ and a coupling $(G^L, G_d, G^U)$ such that $G^L \sim \mathcal{G}(n, p_1)$, $G^U \sim \mathcal{G}(n, p_2)$, $G_d \sim \mathcal{G}(n, d)$ and $\mathbb{P}(G^L \subseteq G_d \subseteq G^U) = 1 - o(1)$.*

For $\log n \ll d \ll n^{1/3}/\log^2 n$, Kim and Vu proved a weakened version of the sandwich conjecture where $G_d \subseteq G^U$ is replaced by $\Delta(G^U \setminus G_d)$ being small (see [8, Theorem 2]). Note that this sandwich conjecture, even the weakened version, allows direct translation of many results from $\mathcal{G}(n, p)$ to $\mathcal{G}(n, d)$, including the chromatic number, the Hamiltonicity, etc., which are well studied in $\mathcal{G}(n, p)$.

Recently, Dudek, Frieze, Ruciński and M. Šileikis [4] proved a weakened sandwich conjecture for random hypergraphs where $d = o(n)$. In this weakened version the condition $G_d \subseteq G^U$ is removed. Their result allows translation of all monotonely increasing properties from $\mathcal{G}(n, p)$ and $\mathcal{G}(n, d)$. For instance, they applied it to prove the Hamiltonicity of random regular hypergraphs.

An immediate corollary of the sandwich conjecture, if it were true, is that, one can couple two random regular graphs $G_{d_1} \sim \mathcal{G}(n, d_1)$ and $G_{d_2} \sim \mathcal{G}(n, d_2)$ such that a.a.s. $G_{d_1} \subseteq G_{d_2}$, if $d_2$ is sufficiently greater than $d_1$. In fact we conjecture that such a coupling exists as long as $d_2 \geq d_1$. However, the weakened versions of the sandwich conjecture, as proved in [8] and [4], are not strong enough to imply the existence of such a coupling, even when $d_2$ is much greater than $d_1$.

**Conjecture 2.** *Let $d_1 \leq d_2$ be integers between 1 and $n - 1$. There exists a coupling $(G_{d_1}, G_{d_2})$ such that $G_{d_1} \sim \mathcal{G}(n, d_1)$, $G_{d_2} \sim \mathcal{G}(n, d_2)$, and $\mathbb{P}(G_{d_1} \subseteq G_{d_2}) = 1 - o(1)$.*

*Remark*: This conjecture or some variate of it has already been speculated and discussed in the community. But we haven't found any written document about it.

In this paper, we confirm the sandwich conjecture for all $d$ where $\min\{d, n - d\} \gg n/\log n$. Moreover, we prove the sandwich conjecture for all degree sequences $\mathbf{d}$ where each coordinates in $\mathbf{d}$ do not deviates too much from $d$. In addition, we confirm Conjecture 2 for degrees $d_1$ and $d_2$ from the corresponding dense range and also for constant $d_1$.

## 2 Main results

Given a sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of (not necessarily nonnegative) integers, let $d_{\max}$ and $d_{\min}$ denote the maximum and minimum components of $\mathbf{d}$ respectively, and denote $d_{\max} - d_{\min}$, a semi-norm of $\mathbf{d}$, by $\|\mathbf{d}\|$. Define near-regular degree sequence as follows.

**Definition 3.** *For $d > 0$, we say a degree sequence $\mathbf{d}$ is $d$-near-regular if $\|\mathbf{d}\| = o(d(n - d)/n)$.*

Let $\mathcal{G}(n,\mathbf{d})$ denote the uniform probability space over the set of graphs on $[n]$ with degree sequence $\mathbf{d}$. Note that for $\mathcal{G}(n,\mathbf{d})$ to be non-empty, $\mathbf{d}$ necessarily has nonnegative coordinates and even sum. All asymptotics in the paper refer to $n\to\infty$. For two sequences of real numbers $a_n$ and $b_n$, we say $a_n=o(b_n)$ if $b_n\neq 0$ eventually and $\lim_{n\to\infty} a_n/b_n=0$. We say $a_n=O(b_n)$ if there exists a constant $C>0$ such that $|a_n|\le C|b_n|$ for all $n$. We use notation $a_n=\omega(b_n)$ and $a_n=\Omega(b_n)$ to denote $b_n=o(a_n)$ and $b_n=O(a_n)$ respectively. Alternatively, we write $a_n\ll b_n$ and $a_n\gg b_n$ for $a_n=o(b_n)$ and $a_n=\omega(b_n)$, if both $a_n$ and $b_n$ are positive sequences. For convenience, $\Omega(a_n)$ and $\omega(a_n)$ in our paper only refer to positive sequences, e.g. $\Omega(a_n)$ denotes some sequence $b_n$ such that $b_n>0$ always and $b_n=\Omega(a_n)$. We use $a_n=\Theta(b_n)$ to denote $a_n=O(b_n)$ and $b_n=O(a_n)$.

In the following we list some of our main results.

**Theorem 4.** *Assume* $\min\{d,n-d\}=\omega(n/(\log n))$ *and assume that* $\mathbf{d}$ *is* $d$*-near-regular. For any* $\xi$ *satisfying*

$$\sqrt{\left(\frac{(1-d/n)\ln\left(\frac{1}{1-d/n}\right)}{d/n}+\ln(n/d)\right)\frac{\|\mathbf{d}\|}{d}}+\frac{n}{d\log n}\ll\xi, \tag{1}$$

*there is a multiple coupling* $(G^L,G_{\mathbf{d}},G^U)$ *such that* $G^L\sim\mathcal{G}(n,p_1)$, $G^U\sim\mathcal{G}(n,p_2)$, $G_{\mathbf{d}}\sim\mathcal{G}(n,d)$, *where* $p_1=(1-\xi)d/n$, $p_2=(1+\xi)d/n$, *and*

$$\mathbb{P}(G^L\subseteq G_{\mathbf{d}}\subseteq G^U)=1-\exp(-\Omega(n/(\log n)^4)).$$

**Corollary 5.** *Assume* $n/\log n\ll d_1\le d_2$ *and* $n-d_2\gg n/\log n$, *and* $d_2-d_1=\omega(n/\log n)$. *Then there is a coupling* $(G_{d_1},G_{d_2})$ *such that* $G_{d_1}\sim\mathcal{G}(n,d_1)$, $G_{d_2}\sim\mathcal{G}(n,d_2)$ *and*

$$\mathbb{P}(G_{d_1}\subseteq G_{d_2})=1-\exp(-\Omega(n/(\log n)^4)).$$

Note that the sandwich conjecture is still open for $d=O(n/\log n)$, since both results in [8] and [4] did not prove the existence of a coupling between $\mathcal{G}(n,d)$ and $\mathcal{G}(n,p)$, $p=(1+\varepsilon)d/n$, where the former is contained in the latter with high probability. In the next theorem, we prove the existence of such a coupling when $d$ is fixed and $p$ is rather large.

**Theorem 6.** *Let* $d>0$ *be fixed and let* $p=\omega(n^{-1/d}\log^{1/d}n)$. *Then, there is a coupling* $(G_d,G)$ *such that* $G_d\sim\mathcal{G}(n,d)$, $G\sim\mathcal{G}(n,p)$ *and*

$$\mathbb{P}(G_d\subseteq G)=1-o(1).$$

**Corollary 7.** *Let* $d>0$ *be fixed. Assume* $\Delta=\omega(n^{1-1/d}\log^{1/d}n)$. *Then, there is a coupling* $(G_d,G_\Delta)$ *such that* $G_d\sim\mathcal{G}(n,d)$, $G_\Delta\sim\mathcal{G}(n,\Delta)$ *and*

$$\mathbb{P}(G_d\subseteq G_\Delta)=1-o(1).$$

The proofs are based on new asymptotic enumeration theory developed in [7].

# References


[1] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European J. Combin.* 1 (1980), 311–316.

[2] C. Cooper, A. Frieze and B. Reed, Random regular graphs of non-constant degree: connectivity and Hamiltonicity, *Combinatorics, Probability and Computing* 11, no. 3 (2002): 249–261.

[3] D. Dellamonica, Y. Kohayakawa, V. Rödl, A. Ruciński. An improved upper bound on the density of universal random graphs, *Random Structures & Algorithms* 46(2), (2015), 274–299.

[4] A. Dudek, A. Frieze, A. Ruciński, and M. Šileikis, Embedding the Erdős-Rényi hypergraph into the random regular hypergraph and Hamiltonicity, *Journal of Combinatorial Theory, Series B* 122 (2017): 719–740.

[5] P. Erdős and A. Rényi, On random graphs. I. *Publ. Math. Debrecen* 6 1959 290–297.

[6] P. Erdős and A. Rényi, On the evolution of random graphs, *Bull. Inst. Internat. Statist.* 38 (1961) 343–347.

[7] M. Isaev and B. D. McKay, Complex martingales and asymptotic enumeration, *Random Structures & Algorithms*, to appear (accepted December 2017), DOI: 10.1002/rsa.20754.

[8] J. H. Kim and V. H. Vu, Sandwiching random graphs: universality between random graph models, *Advances in Mathematics* 188.2 (2004): 444–469.

[9] M. Krivelevich, B. Sudakov, V. H. Vu and N. C. Wormald, Random regular graphs of high degree, *Random Structures & Algorithms* 18, no. 4 (2001): 346–363.