A rainbow blow-up lemma

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Abstract

We prove a rainbow version of the blow-up lemma of Komlós, Sárközy and Szemerédi for μn-bounded edge colourings. This enables the systematic study of rainbow embeddings of bounded degree spanning subgraphs. As one application, we show how our blow-up lemma can be used to transfer the bandwidth theorem of Böttcher, Schacht and Taraz to the rainbow setting. Our proof methods include the strategy of an alternative proof of the blow-up lemma given by Rödl and Ruciński, the switching method, and the partial resampling algorithm developed by Harris and Srinivasan.

A subgraph H of an edge-coloured graph G is called *rainbow* if all its edges have different colours. Rainbow colourings appear for example in canonical Ramsey theory, and many open problems in combinatorics such as the Ryser–Brualdi–Stein conjecture on partial transversals in Latin squares and the graceful labelling conjecture can be phrased as rainbow subgraph problems. The central question is under which conditions on G and its edge colouring a rainbow copy of H is guaranteed. Here, H is usually a spanning subgraph such as a perfect matching [8, 12, 14, 21], Hamilton cycle [1, 2, 4, 7, 9, 11], spanning tree [3, 10, 22], or a general bounded degree graph [5, 13, 23].

Clearly, a necessary condition for the existence of a rainbow copy of H in G is that H is at least a subgraph of G. Thus, the best one can hope for is to find a rainbow copy of H in G 'whenever' H is a subgraph of G. The blow-up lemma of Komlós, Sárközy and Szemerédi is a powerful tool to find spanning subgraphs, which, since its invention roughly 20 years ago, has significantly shaped the landscape of extremal combinatorics [6, 15, 16, 17, 18, 19]. It is tailored to be used after an application of Szemerédi's regularity lemma and roughly says that super-regular pairs behave like complete bipartite graphs in terms of embedding bounded degree subgraphs. In the present paper, we prove a rainbow blow-up lemma. As one application, we transfer the bandwidth theorem of Böttcher, Schacht and Taraz [6] to the rainbow setting. It would be interesting to find out whether other results can be transferred in a similar way.

In many of the classical rainbow problems, the host graph G is complete. For instance, Erdős and Stein asked for the maximal k such that any k-bounded edge colouring of K_n contains a rainbow Hamilton cycle (cf. [9]). Here, an edge colouring is *k-bounded* if each colour appears on at most k edges. After several subsequent improvements, Albert, Frieze, and Reed [1] showed that $k = \Omega(n)$, i.e. there exists a constant $\mu > 0$ such that for any μn-bounded edge colouring of K_n, there exists a rainbow Hamilton cycle. Note that this is best possible up to the value of the constant μ. A natural generalization is to ask for general rainbow (spanning) subgraphs. For example, Frieze and Krivelevich [10] showed that there exists some $\mu > 0$ such that any almost spanning tree with bounded degree is contained as a rainbow copy in K_n for any μn-bounded edge colouring of K_n. This was greatly improved by Böttcher, Kohayakawa, and Procacci [5], who showed the following very general result. Given any $n/(51\Delta^2)$-bounded edge colouring of K_n and any graph H on n vertices with $\Delta(H) \leq \Delta$, one can find a rainbow copy of H. Their proof is based on the Lopsided Lovász local lemma as well as the framework of Lu and Székely [20] for random injections. Using these tools, they show that a random injection $V(H) \to V(K_n)$ yields with positive probability a rainbow copy of H. Kamčev, Sudakov, and Volec [13] recently extended this result to the setting

where G is complete multipartite, and Sudakov and Volec [23] considered the case when the number of cherries in H is restricted (instead of the maximum degree).

There is a major stumbling stone if one wants to consider the above problem for incomplete host graphs, say, for example quasi-random graphs with density d for some arbitrarily small (fixed) $d > 0$. If G is complete, then any injection $V(H) \to V(G)$ yields a valid embedding of H (similarly for the multipartite setting). However, this is not the case for general host graphs G, where a random injection yields a valid embedding with exponentially small probability. Restricting the probability space to the 'valid' injections does not seem to work with the Lu–Székely framework, as the latter relies on the perfect symmetry of the setup.

Some recent results on rainbow subgraphs in incomplete host graphs with μn-bounded edge colourings were obtained using the so-called 'switching method'. For example, Coulson and Perarnau [8] found rainbow perfect matchings in Dirac bipartite graphs, improving an approximate result of [7]. The crucial property is that given a perfect matching M and an edge $e \in M$ (which is in conflict with another edge in M), there are many ways of 'switching e out of M' to obtain a new perfect matching which does not contain e. As another example, in an upcoming paper, Coulson, Keevash, Perarnau, and Yepremyan show the existence of a rainbow F-factor in a graph G whenever $\delta(G) \geq (\delta_F + o(1))n$, where δ_F is the minimum degree threshold for the existence of an F-factor (cf. [18]). (Here F is an arbitrary fixed graph.) However, the switching method seems to be limited to 'simply structured' spanning graphs H with rich symmetry properties.

We are motivated by the following question:

> *Given a (dense) graph G on n vertices with a μn-bounded edge colouring and a (bounded degree) graph H on n vertices, is there a rainbow copy of H in G?*

By proving a rainbow blow-up lemma, we provide a tool which allows for the systematic study of this question, profiting from various techniques and methods which have been developed in the non-coloured setting. In particular, we give affirmative answers to the above question if G is quasi-random or has sufficiently high minimum degree.

We remark that the constant $\mu > 0$ we obtain is very small. Nevertheless, our rainbow blow-up lemma can also serve as a useful tool to study problems where the edge colouring is only proper, say, but not globally bounded. To appreciate this, note that if one chooses a random subset U of vertices of size μn, then with high probability, the colouring restricted to this subset is $\mu|U|$-bounded, and the blow-up lemma could be applied to U to complete a partial embedding (constructed outside U using different methods).

We will discuss the blow-up lemma in more detail in the next subsection. As mentioned above, commonly used techniques like the Lu–Székely framework and the switching method do not seem capable of dealing with quasi-random host graphs G and/or general graphs H. (Here, the idea would have been to use the original blow-up lemma as a 'blackbox' result.) Another natural question is whether the proof of the blow-up lemma can be adapted to work in the rainbow setting. In the original proof due to Komlós, Sárközy and Szemerédi, the vertices of H are embedded one by one using a randomized algorithm, until all but a small fraction of the vertices are embedded, and the embedding is then completed using the König-Hall theorem. Note that this approach is extremely vulnerable in the rainbow setting, as already a constant number of choices can render the algorithm unsuccessful (as a vertex may be incident to only a constant number of different colours), which seems rather hopeless.

To overcome these obstacles, several new ideas are needed. As an underlying strategy, we employ the alternative proof of the blow-up lemma given by Rödl and Ruciński, and combine it

with various techniques such as the partial resampling algorithm, the switching method, and a parallelization of the embedding procedure.

In order to state a simplified version of our rainbow blow-up lemma, we need some more terminology. For $k \in \mathbb{N}$, we write $[k]_0 := [k] \cup \{0\} = \{0, 1, \dots, k\}$. We say that $(H, G, R, (X_i)_{i\in[r]_0}, (V_i)_{i\in[r]_0})$ is a *blow-up instance* if the following hold:

- H and G are graphs, $(X_i)_{i\in[r]_0}$ is a partition of $V(H)$ into independent sets, $(V_i)_{i\in[r]_0}$ is a partition of $V(G)$, and $|X_i| = |V_i|$ for all $i \in [r]_0$;
- R is a graph on $[r]$ such that for all distinct $i, j \in [r]$, the graph $H[X_i, X_j]$ is empty if $ij \notin E(R)$.

Here, X_0 and V_0 are so-called 'exceptional sets'. For simplicity, we assume in this subsection that they are empty.

For a graph G and two disjoint subsets $S, T \subseteq V(G)$, denote by $e_G(S, T)$ the number of edges of G with one endpoint in S and the other one in T, and define

$$d_G(S, T) := \frac{e_G(S, T)}{|S||T|}$$

as the *density* of the pair S, T in G. We say that the bipartite graph $G[V_1, V_2]$ is *lower* (ϵ, d)*-super-regular* if

- for all $S \subseteq V_1$ and $T \subseteq V_2$ with $|S| \geq \epsilon|V_1|$, $|T| \geq \epsilon|V_2|$, we have $d_G(S, T) \geq d - \epsilon$;
- for all $i \in [2]$ and $v \in V_i$, we have $|N_G(v) \cap V_{3-i}| \geq (d - \epsilon)|V_{3-i}|$.

We say that the blow-up instance $(H, G, R, (X_i)_{i\in[r]_0}, (V_i)_{i\in[r]_0})$ is *lower* (ϵ, d)*-super-regular* if for all $ij \in E(R)$, the bipartite graph $G[V_i, V_j]$ is lower (ϵ, d)-super-regular.

We now state a simplified version of the rainbow blow-up lemma. The full statement also allows exceptional vertices and candidate sets. Moreover, it does not only apply to rainbow embeddings, but to slightly more general conflict-free embeddings.

Lemma 1 (Rainbow blow-up lemma—simplified)**.** *For all d, Δ, r, there exist $\epsilon = \epsilon(d, \Delta)$, $\mu = \mu(d, \Delta, r) > 0$ and an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$.*

Let $(H, G, R, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ be a lower (ϵ, d)-super-regular blow-up instance. Assume further that

(i) $\Delta(H) \leq \Delta$,

(ii) $|V_i| = (1 \pm \epsilon)n/r$ *for all* $i \in [r]$.

Then, given any μn-bounded edge colouring of G, there exists a rainbow embedding of H into G (where X_i is mapped to V_i for all $i \in [r]$).

Observe that ϵ does not depend on r, which is crucial in applications. Originally the blow-up lemma was formulated with R being the clique on at most Δ vertices. In essentially all applications it is applied to many clique blow-ups iteratively. However, in order to have a useful rainbow blow-up lemma, we cannot apply it independently to two or more clique blow-ups as in both parts the blow-up lemma may use the same colour.

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