Square-free graphs with no six-vertex induced path

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Abstract

We elucidate the structure of (P_6, C_4)-free graphs by showing that every such graph either has a clique cutset, or a universal vertex, or belongs to several special classes whose structure is completely characterized. Using these results we show that every such graph G has chromatic number at most $\lceil \frac{5}{4}\omega(G) \rceil$ and that this is the best possible bound. Another consequence is the existence of polynomial-time algorithms that compute the chromatic number and the stablity number of any graph in this class.

1 Introduction

For any integer k, a *k-coloring* of a graph G is a mapping $c : V(G) \to \{1, \ldots, k\}$ such that any two adjacent vertices u, v in G satisfy $c(u) \neq c(v)$. A graph is *k-colorable* if it admits a k-coloring. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G is k-colorable. Determining whether a graph is k-colorable is NP-complete for every fixed $k \geq 3$ [2, 6].

Given a family of graphs $\mathcal{F}$, a graph G is *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{F}$; when $\mathcal{F}$ has only one element F we say that G is F-free. For any integer ℓ we let P_ℓ denote the path on ℓ vertices and C_ℓ denote the cycle on ℓ vertices. Here we focus on (P_6, C_4)-free graphs.

A *clique* in a graph G is a set of pairwise adjacent vertices. Let $\omega(G)$ denote the maximum clique size in a graph G. Following Gyárfás [5], we say that a class of graphs is *χ-bounded* if there is a function f such that every member G of the class satisfies $\chi(G) \leq f(\omega(G))$. Gyárfás [5] conjectured that if T is any tree (or forest) then the class of T-free graphs is χ-bounded; moreover he proved the conjecture when T is a path. Gravier et al. [4] improved Gyárfás's bound slightly by proving that every P_k-free graph G satisfies $\chi(G) \leq (k-2)^{\omega(G)-1}$. In particular every P_6-free graph G satisfies $\chi(G) \leq 4^{\omega(G)-1}$. One may wonder whether this exponential bound can be improved.

Gaspers and Huang [3] studied the (P_6, C_4)-free graphs and showed that every such graph G satisfies $\chi(G) \leq \frac{3}{2}\omega(G)$. We improve their result and establish the best possible bound, as follows.

Theorem 1.1. *Let G be any (P_6, C_4)-free graph. Then $\chi(G) \leq \lceil \frac{5}{4}\omega(G) \rceil$. Moreover, this bound is tight.*

One can readily see that the bound is tight on the following example (as in [1]). Let G be a graph whose vertex-set is partitioned into five cliques $Q_1, \ldots, Q_5$ such that for each i mod 5, every vertex in Q_i is adjacent to every vertex in $Q_{i+1} \cup Q_{i-1}$ and to no vertex in $Q_{i+2} \cup Q_{i-2}$, and $|Q_i| = q$ for all i ($q > 0$). It is easy to check that G is (P_6, C_4)-free. Moreover, we have $\omega(G) = 2q$, and $\chi(G) \geq \lceil \frac{5}{2}q \rceil$ since G has no stable set of size 3. Hence $\chi(G) \geq \lceil \frac{5}{4}\omega(G) \rceil$.

Theorem 1.2. *There is a polynomial-time algorithm which computes the chromatic number of any (P_6, C_4)-free graph.*

Theorem 1.3. *There is a polynomial-time algorithm which computes the stability number of any (P_6, C_4)-free graph.*

Theorems 1.1, 1.2 and 1.3 will be derived from the structural theorem below (Theorem 3.1).

2 Some special graphs

Let F_1, F_2, F_3 be three graphs defined as follows, as in [3].

The graph F_1 has eight vertices $v_1, ..., v_6, a, b$ such that $v_1, \ldots, v_6$ induce a 6-cycle with edges $v_i v_{i+1}$ (mod 6), and $N_{F_1}(a) = \{v_1, v_2, v_3, b\}$ and $N_{F_1}(b) = \{v_3, v_4, v_5, a\}$.

The graph F_2 has eight vertices $v_1, ..., v_5, a, b, c$ such that $v_1, \ldots, v_5$ induce a 5-cycle with edges $v_i v_{i+1}$ (mod 5), and $N_{F_2}(a) = \{v_1, v_4, v_5, b, c\}$, $N_{F_2}(b) = \{v_1, v_2, a\}$ and $N_{F_2}(c) = \{v_3, v_4, a\}$.

The graph F_3 has nine vertices $v_1, \ldots, v_6, x, y, z$ such that $v_1, \ldots, v_6$ induce a 6-cycle with edges $v_i v_{i+1}$ (mod 6), and $N_{F_3}(x) = \{v_1, v_2, v_3, y, z\}$, $N_{F_3}(y) = \{v_3, v_4, v_5, x, z\}$, and $N_{F_3}(z) = \{v_5, v_6, v_1, x, y\}$.

Let H_1 be the Petersen graph.

Let H_2 have vertices $v_1, ..., v_6, a, b, c$ such that $v_1, ..., v_6$ induce a C_6, with edges $v_i v_{i+1}$ (modulo 6), and $N_{H_2}(a) = \{v_1, v_2, v_3, v_6, b, c\}$, $N_{H_2}(b) = \{v_3, v_4, v_5, v_6, a, c\}$, and $N_{H_2}(c) = \{v_3, v_6, a, b\}$.

Let H_3 have vertices $v_1, ..., v_9$ with edges $v_i v_{i+1}$ and $v_i v_{i+2}$ for all i (mod 9).

Let H_4 have vertices $v_1, ..., v_6, a, b, c$ such that $v_1, ..., v_6$ induce a C_6 with edges $v_i v_{i+1}$ for all i (mod 6), and $N_{H_4}(a) = \{v_6, v_1, v_2, v_3, b\}$, $N_{H_4}(b) = \{v_1, v_2, v_3, v_4, a, c\}$, and $N_{H_4}(c) = \{v_2, v_3, v_4, v_5, b\}$.

Let H_5 have vertices $v_1, ..., v_5, t_1, ..., t_5$ such that $v_1, ..., v_5$ induce a C_5 with edges $v_i v_{i+1}$ for all i (mod 5) and $N_{H_5}(t_i) = \{v_{i-1}, v_i, v_{i+1}\}$ for all i.

For integers $k, \ell \geq 0$ let $F_{k,\ell}$ be the graph whose vertex set can be partitioned into sets A, B, U, W and $\{x, y, z\}$ such that:

- $A = \{a_0, a_1, \ldots, a_k\}$ is a clique of size $k+1$, and $U = \{u_1, u_2, \ldots, u_k\}$ is a stable set of size k, and the edges between A and U form a matching of size k, namely, $[A, U] = \{a_i u_i \mid i \in \{1, \ldots, k\}\}$;
- $B = \{b_0, b_1, \ldots, b_\ell\}$ is a clique of size $\ell + 1$, and $W = \{w_1, \ldots, w_\ell\}$ is a stable set of size ℓ, and the edges between B and W form a matching of size ℓ, namely, $[B, W] = \{b_j w_j \mid j \in \{1, \ldots, \ell\}\}$;
- The neighborhood of x is $A \cup U \cup W \cup \{y\}$;
- The neighborhood of y is $B \cup U \cup W \cup \{x\}$;
- The neighborhood of z is $A \cup B$.

In a graph G, let A, B be disjoint subsets of $V(G)$. It is easy to see that the following two conditions (i) and (ii) are equivalent: (i) any two vertices $a, a' \in A$ satisfy either $N_B(a) \subseteq N_B(a')$ or $N_B(a') \subseteq N_B(a)$; (ii) any two vertices $b, b' \in B$ satisfy either $N_A(b) \subseteq N_A(b')$ or $N_A(b') \subseteq N_A(b)$. If this condition holds we say that the pair $\{A, B\}$ is *graded*. It is easy to see that in a C_4-free graph any two disjoint cliques form a graded pair.

Blowups A *blowup* of a graph H is any graph G such that $V(G)$ can be partitioned into $|V(H)|$ (not necessarily non-empty) cliques Q_v, $v \in V(H)$, such that $[Q_u, Q_v]$ is complete if $uv \in E(H)$, and $[Q_u, Q_v] = \emptyset$ if $uv \notin E(H)$.

Bands A *band* is any graph G whose vertex-set can be partitioned into seven sets $Q_1, ..., Q_5, R_2, R_3$ such that:

- Each of $Q_1, ..., Q_5, R_2, R_3$ is a clique.

- The sets $[Q_5, Q_1 \cup Q_4]$, $[R_2, Q_1 \cup Q_2 \cup Q_3]$, $[R_3, Q_2 \cup Q_3 \cup Q_4]$ and $[Q_2, Q_3]$ are complete.
- The sets $[Q_1, Q_3 \cup R_3 \cup Q_4]$, $[Q_4, Q_1 \cup Q_2 \cup R_2]$ and $[Q_5, Q_2 \cup R_2 \cup Q_3 \cup R_3]$, are empty.
- The pairs $\{Q_1, Q_2\}$, $\{Q_3, Q_4\}$ and $\{R_2, R_3\}$ are graded.

Belts A *belt* is any (P_6, C_4, C_6)-free graph G whose vertex-set can be partitioned into seven sets $Q_1, ..., Q_5, R_2, R_3$ such that:

- Each of $Q_1, ..., Q_5$ is a clique.
- Q_1 is complete to $Q_2 \cup R_2 \cup Q_5$ and anticomplete to $Q_3 \cup R_3 \cup Q_4$, and Q_4 is complete to $Q_3 \cup R_3 \cup Q_5$ and anticomplete to $Q_2 \cup R_2 \cup Q_1$, and Q_5 is anticomplete to $Q_2 \cup R_2 \cup Q_3 \cup R_3$.
- For each $j \in \{2, 3\}$, Q_j is complete to R_j, and no vertex of R_j is universal in $G[R_j]$.
- For each $j \in \{2, 3\}$, every vertex in $Q_j \cup R_j$ has a neighbor in $Q_{5-j} \cup R_{5-j}$.

Boilers A *boiler* is a (P_6, C_4, C_6)-free graph G whose vertex-set can be partitioned into five sets Q, A, B, L, M such that:

- Q, A and B are cliques;
- $[Q, A]$, $[Q, M]$, and $[B, L]$ are complete;
- $[Q, B]$, $[Q, L]$ and $[L, M]$ are empty;
- Every vertex in L has a neighbor in A;
- For some integer $k \geq 3$, M is partitioned into k non-empty sets $M_1, ..., M_k$, and B is partitioned into k non-empty sets $B_1, ..., B_k$, such that for each $i \in \{1, ..., k\}$ every vertex in M_i has a neighbor in B_i and no neighbor in $B \setminus B_i$;
- A is complete to $M_1 \cup M_2 \cup B_1 \cup B_2$, and for each $i \in \{3, ..., m\}$ every vertex in A is either complete or anticomplete to $M_i \cup B_i$.

We consider that the definition of blowups and of bands is also a complete description of the structure of such graphs. However this is not so for belts and boilers. Belts and boilers have many additional technical properties that are useful but too lengthy to describe here. Such properties can be used to give a precise description of the structure of belts and boilers.

3 The main result

Theorem 3.1. *If G is any (P_6, C_4)-free graph, then one of the following holds:*

- *G has a clique cutset.*
- *G has a universal vertex.*
- *G is a blowup of either $H_1, ..., H_5$, F_3 or $F_{k,\ell}$ (for some $k, \ell \geq 1$).*
- *G is a band, a belt, or a boiler.*

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