On Weakly Distinguishing Graph Polynomials

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Abstract

A univariate graph polynomial $P(G;X)$ is weakly distinguishing if for almost all finite graphs G there is a finite graph H with $P(G;X) = P(H;X)$. We show that the clique polynomial and the independence polynomial are weakly distinguishing. Furthermore, we show that generating functions of induced subgraphs with property C are weakly distinguishing provided that C is of bounded tree-width. The same holds for the harmonious chromatic polynomial.

Let P be a graph polynomial. A graph G is *P-unique* if every graph H with $P(G;X) = P(H;X)$ is isomorphic to G. A graph H *is a P-mate* of G if $P(G;X) = P(H;X)$ but H is not isomorphic to G. In [10] P-unique graphs are studied for the Tutte polynomial $T(G;X,Y)$, the chromatic polynomial $\chi(G;X)$, the matching polynomial $m(G;X)$ and the characteristic polynomial $char(P;X)$.

A statement holds for almost all graphs if the proportion of graphs of order n for which it holds, tends to 1, when n tends to infinity. A graph polynomial P is *almost complete* if almost all graphs G are P-unique, and it is *weakly distinguishing* if almost all graphs G have a P-mate. In [3] it is conjectured that almost all graphs are χ-unique, and hence T-unique, in other words, both $\chi(G;X)$ and $T(G;X,Y)$ are almost complete. There are plenty of trivial graph polynomials which are weakly distinguishing, like $X^{|V(G)|}$ or $X^{|E(G)|}$. However, one might expect that the prominent graph polynomials from the literature are not weakly distinguishing. Here we show that various non-trivial graph polynomials are still weakly distinguishing.

For a vertex $v \in V(G)$ of a graph G, let $d_G(v)$ denote the degree of v in G. The degree polynomial $Dg(G;X)$ of a graph G is defined as $Dg(G;X) = \sum_{v \in V(G)} X^{d_G(v)}$. A graph G is Dg-unique, also called in the literature a *unigraph*, if it is determined by its degree sequence. An updated discussion on how to recognize unigraphs can be found in [2].

A simple counting argument gives:

Theorem 1. *Almost all graphs G have a Dg-mate.*

For a graph $G = (V(G), E(G))$ and $A \in V(G)$ we denote by $G[A]$ the subgraph of G induced by A. Let $\mathcal{E}$ be the class of edgeless graphs and $\mathcal{C}$ be the class of complete graphs.
The independence polynomial of a graph is defined as $Ind(G;X) = \sum_{A \in V(G): G[A] \in \mathcal{E}} X^{|A|}$.
The clique polynomial of a graph is defined as $Cl(G;X) = \sum_{A \in V(G): G[A] \in \mathcal{C}} X^{|A|}$.
Both were first studied in [7]. For a more recent survey on the independence polynomial see [8].

Theorem 2. *Almost all graphs G have an Ind-mate and a Cl-mate.*

The proof uses estimates for the independence number $\alpha(G)$ and the clique number $\omega(G)$ for random graphs, (see [4] and [6]) together with a counting argument.

This theorem can be generalized: Let $\mathcal{C}$ be a graph property. We say that a function $f : \mathbb{N} \to \mathbb{N}$ is an *independence (clique) function for $\mathcal{C}$* if for every graph $G \in \mathcal{C}$, the graph G has an independent set (clique) of size $f(|V(G)|)$.

Denote by $\hat{\mathcal{C}}$ the class of complement graphs $\bar{G}$ of graphs $G \in \mathcal{C}$, and $P_{\mathcal{C}}(G;X) = \sum_{A \in V(G):G[A] \in \mathcal{C}} X^{|A|}$, and $P_{\hat{\mathcal{C}}}(G;X) = \sum_{A \in V(G):G[A] \in \hat{\mathcal{C}}} X^{|A|}$.

Theorem 3. *Let $\mathcal{C}$ be a graph property that has an independence (clique) function f that satisfies that for all $n \in \mathbb{N}$, $f(n) \geq n/a$ for some fixed $a \in \mathbb{N}$. Then $P_{\mathcal{C}}$ is weakly distinguishing.*

This applies to the following cases:

(i) A graph G is k-degenerate if every induced subgraph of G has a vertex of degree at most k. It is easy to see that every k-degenerate graph G of order n has an independent set of size $\left\lceil \frac{n}{k+1} \right\rceil$.

(ii) Among the k-degenrate graphs we find the graphs of tree-with atmost k, graphs of degree at most k, and planar graphs.

(iii) A k-colorable graphs G has an independent set of size at least $\lceil \frac{n}{k} \rceil$.

(iv) Let $\mathcal{C}$ be a graph property. A function $\gamma : V(G) \to [k]$ is a *$\mathcal{C}$-coloring* if every color class induces a graph in $\mathcal{C}$. Such coloring were studied in [GH83]. If we assume that $\mathcal{C}$ has an independence (clique) function $g(n)$, then the graphs which are $\mathcal{C}$-colorable with at most k colors have an independence (clique) function $f(n) = \left\lceil \frac{g(n)}{k} \right\rceil$.

Therefore, for $\mathcal{C}$ one of the properties above, the graph polynomials $P_{\mathcal{C}}(G;X)$ are all weakly distinguishing.

A harmonious coloring of G with at most k colors is a proper coloring of G such that every pair of colors occurs at most once along an edge. Let $\chi_{harm}(G;k)$ count the number of harmonious colorings of G. It was observed in [9] that $\chi_{harm}(G;k)$ is a polynomial in k.

Theorem 4. *Almost all graphs G have a χ_{harm}-mate.*

The proof uses an observation from [5]. The same holds for an infinite set of less prominent graph polynomials, but the status of P-uniqueness remains open for $T(G;X,Y), \chi(G;X), m(G;X)$ and $char(G;X)$.

References

[1] M. Aigner and E. Triesch, Realizability and uniqueness in graphs, Discrete Mathematics, 136 (1994), pp. 3-20

[2] A. Borri, T. Calamoneri and R. Petreschi, Recognition of Unigraphs through Superposition of Graphs, Journal of Graph Algorithms and Applications, 15.3 (2011), pp. 323-343

[3] B. Bollobás and L. Pebody and O. Riordan, Contraction–deletion invariants for graphs, Journal of Combinatorial Theory, Series B, 80.2 (2000), pp. 320-345

[4] B. Bollobás and P. Erdős, Cliques in random graphs, Math. Proc. Camb. Phil. Soc. 80 (1976), pp. 419-427

[5] E. Drgas-Burchardt and K. Gibek, Harmonious colorings of graphs, Discrete Applied Mathematics, 217 (2017), pp. 175-184

[6] A.M. Frieze, On the independence number of random graphs, Discrete Mathematics 81 (1990), pp 171-175

[GH83] I. Gutman and F. Harary. Generalizations of the matching polynomial. *Utilitas Mathematicae*, 24:97–106, 1983.

[7] C. Hoede and X. Li, Clique polynomials and independent set polynomials of graphs, Discrete Mathematics, 125 (1994), pp. 219-228

[8] V.E. Levit and E. Mandrescu, The independence polynomial of a graph - a survey, Proceedings of the 1st International Conference on Algebraic Informatics, Aristotle University of Thessaloniki, Thessaloniki, 2005, pp. 233-254

[9] J.A. Makowsky and T. Kotek and B. Zilber On Counting Generalized Colorings, Model Theoretic Methods in Finite Combinatorics, Contemporary Mathematics, 558 (2011), pp. 207-242

[10] M. Noy, On Graphs Determined by Polynomial Invariants, Theoretical Computer Science, 307 (2003), pp. 365-384

[11] Chatterjee, Krishnendu, Rasmus Ibsen-Jensen, and Andreas Pavlogiannis, Optimal Tree-decomposition Balancing and Reachability on Low Treewidth Graphs, IST Austria, 2014

[12] Bodlaender, Hans L. "Treewidth: characterizations, applications, and computations." International Workshop on Graph-Theoretic Concepts in Computer Science. Springer, Berlin, Heidelberg, 2006.