Random colorings of bounded degree graphs

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Abstract

A well-known conjecture in graph coloring is that Glauber dynamics on the set of k-colorings of a graph G on n vertices with maximum degree Δ is rapidly mixing for $k \geq \Delta + 2$. In 1999 Vigoda showed rapid mixing time of a modified version of flip dynamics for $k > 11/6\Delta$ implying polynomial time mixing for Glauber dynamics under the same constraints. This conjecture has attracted a lot of attention in the literature and better results are known for certain classes of graphs. We improve Vigoda's bound for general graphs by showing that there exists a constant $\eta > 0$ such that the Glauber dynamics mixes in polynomial time for $k \geq (11/6 - \eta)\Delta$. Our proof combines path coupling with a new kind of metric we introduce to incorporate a count of the extremal configurations of the chain. This "extremal" metric is easier to analyze than stopping-time-based metrics and we believe will have fruitful applications for bounding the mixing times of other Markov chains.

1 Introduction

Let $G = (V, E)$ be a graph on n vertices with maximum degree Δ, and let $[k]$ denote the set $\{1, 2, \ldots, k\}$. A *k-coloring* of G is an assignment $\sigma : V(G) \to [k]$; we say that a k-coloring is *proper* if no two adjacent vertices receive the same color. Counting the number of proper k-colorings of a graph is a computationally hard problem [7]. Jerrum, Valiant, and Vazirani [4] showed that a nearly uniform sampler gives rise to an approximate enumeration, motivating the question of finding an algorithm to efficiently generate uniformly random colorings of a graph. This question is a central topic in computer science and statistical physics.

To this end, we study the following Markov chain Monte Carlo algorithm known as Glauber dynamics (e.g. see [2]). The *Glauber dynamics for k-colorings* is a Markov chain with state space Ω_0, the set of all proper k-colorings of G, and transitions between states given by recoloring at most one vertex; if $X_t = \sigma$ is the state at time t, then we proceed as follows.

1. Choose u uniformly at random among all vertices in $V(G)$.
2. For all vertices $v \neq u$, let $X_{t+1}(v) = \sigma(v)$.
3. Choose c uniformly at random from $[k]$, if c does not appear among the colors in the neighborhood of u then let $X_{t+1}(u) = c$, otherwise $X_{t+1}(u) = \sigma(u)$.

It is easy to check that the chain is ergodic provided that $k \geq \Delta + 2$. A well-known conjecture in the area is that Glauber dynamics mixes in polynomial time (*rapid mixing*) for every $k \geq \Delta + 2$ (and hence that there exists a polynomial-time nearly uniform sampler of k-colorings for every $k \geq \Delta + 2$). Jerrum [3] showed that for $k \geq 2\Delta + 1$ the mixing time is $O(n \log n)$. Salas and Sokal [6] used Dobrushin's uniqueness criterion to obtain similar results. In 2000, Vigoda [8] obtained an important breakthrough in the area by showing that flip dynamics with certain flip parameters has mixing time $O(n \log n)$ for $k > \frac{11}{6}\Delta$, implying that Glauber dynamics has mixing time $O(n^2 \log n)$ in the same regime. These results actually imply that Glauber dynamics has mixing time $O(n^2)$, for instance see Chapter 14 in [5], and also that the k-state zero temperature anti-ferromagnetic

Potts model on $\mathbb{Z}^d$ lies in the disordered phase when $k > \frac{11}{3}d$. In the last 20 years there have been numerous improvements for particular classes of graphs, but as to the original conjecture, no improvement over Vigoda's bound had appeared.

The main result of this talk is to break the $\frac{11}{6}\Delta$ barrier for general graphs.

Theorem 1. *The Glauber dynamics for k-colorings on a graph on n vertices with maximum degree Δ and $k \geq (11/6 - \eta)\Delta$, with $\eta = 1/10230$, satisfies mixing time*

$$t_{mix} \leq O\left((k \log k) \cdot n^2 \log n\right) .$$

Vigoda [8] proved using the comparison theorem of Diaconis and Saloff-Coste [1] that the mixing time of Glauber dynamics can be upper bounded by the mixing time of flip dynamics with certain flip parameters (with an additional factor of n). Hence, to prove Theorem 1, we prove the following result on flip dynamics using slightly different flip parameters.

Theorem 2. *The flip dynamics for k-colorings on a graph on n vertices with maximum degree Δ and $k \geq (11/6 - \eta)\Delta$, with $\eta = 1/10230$, satisfies mixing time*

$$t_{mix} \leq kn \log 4n .$$

As a corollary of Theorem 1 and Lemma 7 of [8], we find that the k-state zero temperature anti-ferromagnetic Potts model on $\mathbb{Z}^d$ lies in the disordered phase when $k > (\frac{11}{3} - 2\eta)d$.

References

[1] P. Diaconis and L. Saloff-Coste. Comparison theorems for reversible Markov chains. *Ann. Appl. Prob.*, **3**(3), 1993, 696–730.

[2] A. Frieze and E. Vigoda. A survey on the use of Markov chains to randomly sample colourings. *Combinatorics, complexity, and chance*, Oxford Lecture Ser. Math. Appl., Vol. 34. Oxford Univ. Press, Oxford, UK, 2007, 53–71.

[3] M.R. Jerrum. A very simple algorithm for estimating the number of k-colourings of a low-degree graph. *Random Structures and Algorithms* 7(2), 1995, 157–165.

[4] M.R. Jerrum, L.G. Valiant, and V.V. Vazirani. Random generation of combinatorial structures from a uniform distribution. *Theoretical Computer Science* 43, 1986, 169–188.

[5] D. Levin and Y. Peres. Markov Chains and Mixing Times. *American Mathematical Society*, Second Edition, 201

[6] J. Salas and A. Sokal. Absence of phase transition for antiferromagnetic Potts models via the Dobrushin uniqueness theorem. *Journal of Statistical Physics*, 86(3-4), 1997, 551–579.

[7] L. G. Valiant. The complexity of enumeration and reliability problems. *SIAM Journal on Computing*, 8(3), 1979, 410–421.

[8] E. Vigoda. Improved bounds for sampling colorings. *Journal of Mathematical Physics*, 41(3), 2000, 1555–1569.