Triangle-free projective-planar graphs with diameter two: domination and characterization

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Abstract

MacGillivray and Seyffarth (J. Graph Theory 22 (1996)) proved that planar graphs with diameter two have domination number at most three and Goddard and Henning (J. Graph Theory 40 (2002)) proved that there is only one planar graph that attains the upper bound. In this article, we prove that the domination number of triangle-free projective-planar graphs with diameter two is at most three and there are exactly seven graphs that attains the upper bound. Interestingly, six of these seven graphs can be obtained from the famous Grötzsch graph, Wagner graph or Petersen graph by addition or deletion of (specific) vertices. We also generalize a result of Plesník from 1975 and characterize all triangle-free projective-planar graphs with diameter two. Our proof gives a forbidden minor characterization for this class of graphs.

1 Introduction

All graphs considered in this article are undirected and loopless. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph G is triangle-free if G has no induced cycle of order three. In 1975, Plesník [10] characterized all triangle-free graphs of diameter 2 that can be embedded in the plane i.e. planar graphs. This characterization provided by Plesník has been used by researchers [9]. In this article, we focus on characterizing triangle-free graphs with diameter two that can be embedded in the projective plane i.e. non-orientable surface of genus one [8]. Archdeacon [2] and Glover *et al.* [4] proved that there are 103 topologically irreducible graphs for the projective plane and these graphs actually correspond to a set of 35 excluded minors [1, 8]. For more comprehensive surveys on this topic see [3, 8, 6].

In this article, we also study the *domination number* of triangle-free projective-planar graphs with diameter two. For a graph G, if $X, Y \subseteq V(G)$, then we say that X *dominates* Y if every vertex of $Y \setminus X$ is adjacent to some vertex of X. In particular, if X dominates $V(G)$, then X is called a dominating set of G. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. In 1996, MacGillivray and Seyffarth [7] proved that planar graphs with diameter two have domination number at most three and in 2002, Goddard and Henning [5] proved that there is only one planar graph that attains the upper bound (and this graph contains a triangle). They also proved that for each surface $\mathbb{S}$, there are finitely many graphs with diameter two and domination number more than two that can be embedded in $\mathbb{S}$. In this article, we prove that the domination number of triangle-free projective-planar graphs with diameter two is at most three and there are exactly seven graphs that attains the upper bound. Interestingly, six of these seven graphs can be obtained from the famous Grötzsch graph, Wagner graph or Petersen graph (all of them defined later) by addition or deletion of (specific) vertices.

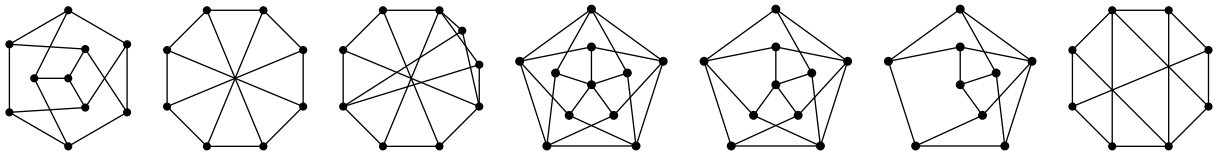


Figure 1: The graph family $\mathcal{D}$. (a) Petersen graph (P); (b) Wagner graph (W); (c) W^+; (d) Grötzsch graph (M); (e) M^-; (f) $M^=$; (g) $K^*_{3,4}$

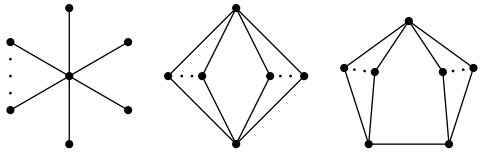


Figure 2: The graph family $\mathcal{P}$.

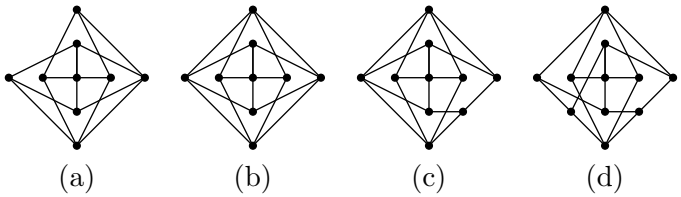


Figure 3: (a) The graph F_6; (b), (c), (d) forms the graph family $\mathcal{D}'$.

2 Our results

Before we state the main theorems, we define some notations and introduce some families of graphs. Let $\mathcal{PP}_4$ denote the class of triangle-free projective-planar graphs with diameter two. For a graph G and a vertex $v \in V(G)$, let $N(v) = \{u \in V(G) \colon uv \in E(G)\}$ denote the *open neighbourhood* of v. Adding a *twin* of v in G means adding a new vertex v' in G such that $N(v) = N(v')$. For an integer n, let C_n denote the cycle of order n. In this article, P shall denote the Petersen graph (Figure 1(a)). The Wagner graph (W) is the graph formed by adding edges between the antipodal vertices of C_8 (Figure 1(b)). Denote by W^+, the graph obtained by adding a twin of an arbitrary vertex of W (Figure 1(c)). The Grötzsch graph (M) is formed by taking a C_5 and adding a twin for each vertex in the C_5 and join all of these vertices (the twins) with another new vertex (Figure 1(d)). The graph M^- is obtained from M by deleting one degree three vertex (Figure 1(e)). The graph $M^=$ is obtained from M^- by deleting a degree three vertex such that the resultant graph has diameter two (Figure 1(f)). The graph $K^*_{3,4}$ is formed from $K_{3,4}$ by deleting a degree four vertex u with $N(u) = \{v_1, v_2, v_3, v_4\}$ and adding two new vertices x and y with $N(x) = \{v_1, v_2, y\}$ and $N(y) = \{v_3, v_4, x\}$ (Figure 1(g)). Let $\mathcal{D}$ be the set $\{P, W, M, W^+, M^-, M^=, K^*_{3,4}\}$.

Theorem 1. *Any graph $G \in \mathcal{PP}_4$ has domination number at most three. Moreover, $\gamma(G) = 3$ if and only if G is one of the graphs in $\mathcal{D}$.*

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ denote the family of graphs shown in Figure 2, where $\mathcal{P}_1 = \{K_{1,n} \colon n \geq 1\}$ (Figure 2(a)), $\mathcal{P}_2 = \{K_{2,n} \colon n \geq 2\}$ (Figure 2(b)), and $\mathcal{P}_3$ denotes all graphs obtained by adding

twins of two nonadjacent vertices of the C_5 (Figure 2(c)). Plesník [10] proved that a graph G of diameter 2 is triangle-free planar if and only if $G \in \mathcal{P}$. The graph $K^t_{m,n}$ is formed by subdividing an edge of the complete bipartite graph $K_{m,n}$ and then adding $t-1$ twins of this new degree two vertex. Let $\mathcal{K}_{SD} = \mathcal{P}_2 \cup \mathcal{P}_3 \cup \left[\bigcup_{t\in\mathbb{N}}(K^t_{3,3} \cup K^t_{3,4})\right]$. Note that $\mathcal{K}_{SD}$ does not contain $K_{3,3}$ and $K_{3,4}$. Define $\mathcal{H} = \mathcal{P}_1 \cup \{K_{3,3}, K_{3,4}\} \cup \mathcal{K}_{SD} \cup \mathcal{D}$. Note that, all graphs in $\mathcal{H}$ can be embedded in the projective plane [11][1] and have diameter two and no triangle. In this article, we prove the following theorem.

Theorem 2. *A graph $G \in \mathcal{PP}_4$ if and only if $G \in \mathcal{H}$.*

By *contraction* of an edge uv in a graph G we replace u and v by a new vertex w adjacent to all of the neighbors of u and v. A graph H is called a *minor* of a graph G if H can be formed from G by deleting edges and vertices and by contracting edges of G. Let $\mathcal{C}$ be a class of graphs and $Forb(\mathcal{C})$ denote the family of graphs that does not contain any graph in $\mathcal{C}$ as minor. Let $K^-_{4,4}$ be the bipartite graph obtained by deleting one edge of $K_{4,4}$. Let F_6 denote the graph shown in Figure 3(a) and $\mathcal{D}'$ denote the set of graphs shown in Figure 3(b), 3(c) and 3(d). Observe that all graphs in $\mathcal{D}'$ contains F_6 as minor. In this article, we characterize a slightly more general class of graphs than $\mathcal{PP}_4$. Specifically, we prove the following theorem.

Theorem 3. *Let G be a triangle-free graph with diameter two and $\mathcal{C} = \{K_{3,5}, K^-_{4,4}\}$. Then $G \in Forb(\mathcal{C})$ if and only if $G \in \mathcal{H} \cup \mathcal{D}'$.*

It is known that the graphs $F_6, K_{3,5}$ and $K^-_{4,4}$ are not projective-planar (see $6^{\text{th}}, 11^{\text{th}}$ and 13^{th} figure in [8, p. 198]). So none of the graphs in $\mathcal{D}'$ are projective-planar. Using Theorem 2 and Theorem 3, we have the following corollary.

Corollary 1. *Let G be a triangle-free graph with diameter two and let $\mathcal{C} = \{K_{3,5}, K^-_{4,4}, F_6\}$. Then G is projective-planar if and only if $G \in Forb(\mathcal{C})$.*

3 Proof sketch

First we shall give a proof sketch for Theorem 1. It is easy to verify using Euler's formula [8] that the minimum degree of a triangle-free projective-planar graph is at most three. Therefore, when a triangle-free projective-planar graph has diameter two, its domination number is at most three. The difficult part is to prove that there are exactly seven graphs that attains this upper bound. Notice that all seven graphs in $\mathcal{D}$ belong to the class $\mathcal{PP}_4$ and have domination number three. Moreover, for any graph $H \in \mathcal{H} \setminus \mathcal{D}$, $\gamma(H) \leq 2$. Now from Theorem 2, the proof of Theorem 1 follows immediately.

From our earlier observations, it follows that $\mathcal{H} \subseteq \mathcal{PP}_4$. To prove the other direction of Theorem 2, notice that the class $\mathcal{PP}_4$ is contained in $Forb(\{K_{3,5}, K^-_{4,4}\})$. Now by Theorem 3, we have that any triangle-free graph H with diameter two belongs to $Forb(\{K_{3,5}, K^-_{4,4}\})$ if and only if $H \in \mathcal{H} \cup \mathcal{D}'$. As noted earlier, none of the graphs in $\mathcal{D}'$ are projective-planar. Hence, it follows that $\mathcal{PP}_4 \subseteq \mathcal{H}$. This completes the proof of Theorem 2.

We give a proof sketch of Theorem 3. In what follows, let $\mathcal{C} = \{K_{3,5}, K^-_{4,4}\}$ and let G be a triangle-free graph with diameter two. If $G \in \mathcal{H} \cup \mathcal{D}'$ then clearly $G \in Forb(\mathcal{C})$. Now assume $G \in Forb(\mathcal{C})$. Depending on the minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, we shall divide

[1] We thank Qianping Gu for providing the code to find the embedding of projective-planar graphs [11].

our proof into four main cases. If $\delta(G) = 1$, then G is precisely $K_{1,n}$, for some $n \in \mathbb{N}$. To see this let v be a vertex with degree one and u be the vertex adjacent to v. As G is triangle-free, $N(u)$ induces an independent set in G. The rest of the proof follows easily from the fact that G has diameter two. In the following lemmas we describe the remaining cases.

Lemma 1. *Let G be a triangle-free graph with diameter two and $\delta(G) = 2$. If $G \in Forb(\mathcal{C})$, then G is isomorphic to a graph in $\mathcal{K}_{SD}$.*

Lemma 2. *Let G be a triangle-free graph with diameter two and $\delta(G) = \Delta(G) = 3$. If $G \in Forb(\mathcal{C})$, then G is isomorphic to a graph in $\{K_{3,3}, W, P\}$.*

Notice that all graphs in $\{K_{3,3}, W, P\}$ are projective-planar. From our proof of Lemma 2, we can also prove the following corollary.

Corollary 2. *Let G be triangle-free 3 regular graph with diameter two. Then G is projective-planar.*

Lemma 3. *Let G be a triangle-free graph with diameter two, $\delta(G) = 3$ and $\Delta(G) \geq 4$. If $G \in Forb(\mathcal{C})$, then either $G \in \mathcal{D}'$ or isomorphic to a graph in $\{K_{3,4}, K^*_{3,4}, M, W^+, M^-, M^=\}$.*

Lemma 1, Lemma 2 and Lemma 3, together completes the proof of Theorem 3.

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