Decomposing a Graph into Shortest Paths with Bounded Eccentricity

Léo Planche — **Université Paris Descartes, Université Paris Diderot**
Etienne Birmelé — **Université Paris Descartes**
Fabien de Montgolfier — **Université Paris Diderot**
Laurent Viennot — **Université Paris Diderot**

Abstract

We introduce the problem of hub-laminar decomposition which generalizes that of computing a shortest path with minimum eccentricity (MESP). Intuitively, it consists in decomposing a graph into several paths that collectively have small eccentricity and meet only near their extremities. We show that a graph having such a decomposition with long enough paths can be decomposed in polynomial time with approximated guaranties on the parameters of the decomposition. Moreover, such a decomposition with few paths allows to compute a compact representation of distances with additive distortion. The problem is related to computing an isometric cycle with minimum eccentricity (MEIC). We also show that having an isometric cycle with small eccentricity is related to the possibility of embedding the graph into a cycle with low distortion.

The goal of this paper is to extend the MESP (Minimum Eccentricity Shortest Path) Problem from [1]. It consists in finding a shortest path k-dominating a graph (every vertex is at distance at most k from that path).

To generalize this problem to more complex underlying structures, we introduce the problem of decomposing a graph into subgraphs with bounded shortest-path eccentricity. More precisely, we introduce the hub-laminar decomposition as a set of paths that k-dominates the graph and meet only near their extremities. To formalize this property, we introduce the notion of *hub*, that is a ball with fixed radius r centered at a path endpoint. The *laminar* associated to a path is the set of nodes k-dominated by the path. Our definition requires that an edge between two nodes belonging to two different laminars must also belong to a hub. A laminar joins therefore two hubs, and laminars meet only inside hubs. The main result of the paper is that computing such a decomposition becomes tractable when hub centers are far enough one from another, or equivalently when paths are long enough. The MESP problem is equivalent to a hub-laminar decomposition with one laminar.

The problem of decomposing a graph into λ laminars that k-dominate the graph is not well defined as there may be several trade-offs of parameters λ and k. However, we show that when laminars are long enough compared to parameters r and k, then all (r, k)-hub-laminar decompositions are equivalent (same global structure) and have closely located hubs (except for hubs of degree two that do not affect the global structure).

From a graph perspective, a very natural generalization of MESP is the problem of finding a minimum eccentricity isometric cycle (MEIC), that is a cycle preserving distances that has minimum eccentricity k. Note that such a cycle can be seen as a hub-laminar decomposition with two laminars and two hubs with degree two. An important motivation for the MESP problem is its relationship with embedding a graph into the line with small multiplicative distortion [1]. We similarly show that the MEIC problem is related to embedding a graph into a circle with low multiplicative distortion, i.e. such that distances in the circle are within a constant factor of distances in the graph. Graph embedding in classical metrics is a well studied problem [3]. Another related subject with abundant literature is that of compactly representing the distances of a graph [4]. We show that a decomposition with few laminars ensures a compact representation of distances with bounded additive distortion.

A full length version of this paper was presented at ISAAC 2017 [5].

Definitions

We consider finite, undirected and *connected* graphs. The vertex and edge sets of a graph G are respectively denoted by $V(G)$ and $E(G)$.

Let $B(u,r) = \{v \in V(G) \mid d(u,v) \leq r\}$ denote the *ball* of radius r centered at u. Given a set of vertices U we set $B(U,r) = \cup_{u \in U} B(u,r)$. Given two sets U and W of vertices, we say that U *k-dominates* W when every vertex in W is at distance at most k from some vertex in U, i.e. $W \subseteq B(U,k)$. We say that U has *eccentricity* k, denoted $ecc(U) = k$, when k is the smallest integer such that $B(U,k) = V(G)$.

Definition 1 (Hub-laminar decomposition)**.** *Consider a connected undirected graph G, two positive integers r and k with $k \leq r$, $H = \{h_1, \ldots, h_q\}$ a set of vertices of G called* hub centers*, and $\mathcal{P} = \{P_1, \ldots, P_p\}$ a set of paths of G called* laminar paths*. A ball $B(h,r)$ with $h \in H$ is called a* hub*, and a set $B(P,k)$ with $P \in \mathcal{P}$ is called a* laminar*. $(H, \mathcal{P})$ is an (r,k)-*hub-laminar decomposition *of G if the following conditions are satisfied:*

1. each laminar links two hubs centers: *the endpoints h, h' of any $P \in \mathcal{P}$ belong to H and for every other hub $h'' \in H \setminus \{h, h'\}$,*
$$B(P,k) \cap B(h'', r+1) = \emptyset$$
2. laminars and hubs dominate G: $V(G) = \bigcup_{h \in H} B(h,r) \cup \bigcup_{P \in \mathcal{P}} B(P,k)$
3. each laminar path is locally a shortest path: *any path $P \in \mathcal{P}$ with endpoints h and h' is a shortest path of $G[B(P,k) \cup B(h,r) \cup B(h',r)]$*
4. laminars meet at hubs only: *for all $i \neq j$ and $uv \in E(G)$ such that $u \in B(P_i,k)$ and $v \in B(P_j,k)$, there is a hub center $h \in H$ such that P_i and P_j both have h as endpoint and $u, v \in B(h,r)$.*

The minimal laminar length *of a decomposition $(H, \mathcal{P})$, denoted ℓ, is the minimal length of the paths in $\mathcal{P}$. Its* laminar size*, denoted λ, is the number of paths in $\mathcal{P}$.*

The hub-laminar decomposition yiels naturally a skeleton, which can be simplified into a *quotient graph*.

Definition 2 (Quotient graph)**.** *Given a graph G and an (r,k)-hub-laminar decomposition $(H, \mathcal{P})$ of G, the* ***quotient*** *of this decomposition is an edge-labeled multigraph with vertex-set H and for each $P \in \mathcal{P}$ with endpoints h, h' there is an edge hh' whose label is the length of P.*

The ***reduced quotient graph*** *of a decomposition $(H, \mathcal{P})$ is the multigraph obtained from its quotient graph by repeatedly removing degree 2 nodes: for every vertex u of the quotient incident with exactly two edges uv and uw with respective labels a and b, u and both edges are removed and a new edge vw is added with label $a + b$. (It is a loop when $v = w$.)*

When the quotient graph is a cycle, it naturally raises the question of computing an isometric cycle as a skeleton of the graph. A cycle C in a graph G is *isometric* if it preserves distances, i.e. $d_C(u,v) = d(u,v)$ for all $u, v \in V(C)$.

The MEIC problem consists in finding an isometric cycle with minimum eccentricity.

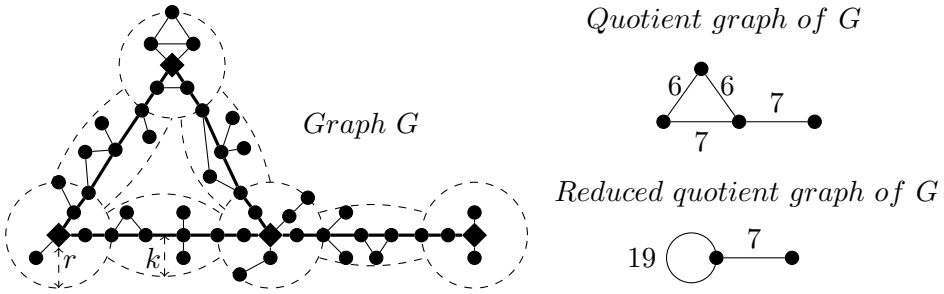


Figure 1: Illustration of an hub-laminar decomposition with $r = 2, k = 1$. Every vertex is at distance r from a hub center (diamond vertices) or at distance k from a laminar path (bold paths between hub centers).

Definition 3 (MEIC)**.** *Given a graph G, find an isometric cycle C such that, for every isometric cycle C', ecc(C) ≤ ecc(C').*

A *circle embedding* of a graph G is a mapping $f : V(G) \to C$ where C is a circle of given length c. It has distortion γ if $d(u, v) \leq d_C(f(u), f(v)) \leq \gamma d(u, v)$ for all u, v in $V(G)$. The *circle distortion* $cd(G)$ of G is the minimum distortion of a circle embedding of G.

A *distance labeling* of a graph G consists in assigning a label L_u to each node $u \in V(G)$ together with a distance estimation function f that outputs an estimation of $d(u, v)$ when given L_u and L_v as input. It has additive distortion α if $d(u, v) \leq f(L_u, L_v) \leq d(u, v) + \alpha$ for all u, v in G.

Results

The MEIC problem can be shown to be NP-complete following a proof similar to one used to show the NP-completeness of the MESP problem [2]. We propose an approximation MEIC algorithm:

Theorem 1. *Given a graph containing a k-dominating isometric cycle with length ℓ, a $6k$-dominating isometric cycle can be computed in $O(n^{4.752} \log(n))$ time. Moreover, the computed cycle is $3k$-dominating when $\ell > 12k + 2$.*

We obtain therefore an algorithm for approximating circle embedding with low distortion.

Theorem 2. *If a graph has circle distortion γ, it is possible to embed it into a circle with distortion $O(\gamma^2)$ in polynomial time.*

Recognizing the general case of an hub-laminar decomposition is not a well defined problem as several decompositions may yield different trade-offs of the parameters. However, when laminars are long enough, all (r, k)-hub-laminar decompositions are indeed $O(k)$ equivalent.

Theorem 3. *Given a graph G having a (r, k)-hub-laminar decomposition $(H, \mathcal{P})$ of minimal laminar length $\ell \geq 10r + 60k + 4$ and integers K, R such that $K \geq 3k$, $R \geq 2K + 3r + 3k$ and $2R + 8K < \ell - 4r - 18k - 4$, it is possible to compute in $O(\min(n, \lambda)m)$ time a (K, R)-hub-laminar decomposition whith the same quotient graph as $(H, \mathcal{P})$.*

From the graph metric point of view, we obtain then a compact representation of distances :

Theorem 4. *Given a graph G having an (r, k)-hub-laminar decomposition with laminar size λ, it is possible to compute in polynomial time a $O(\max\{k, r\})$-additive distance labeling with $O(\lambda \log n)$ bit labels.*

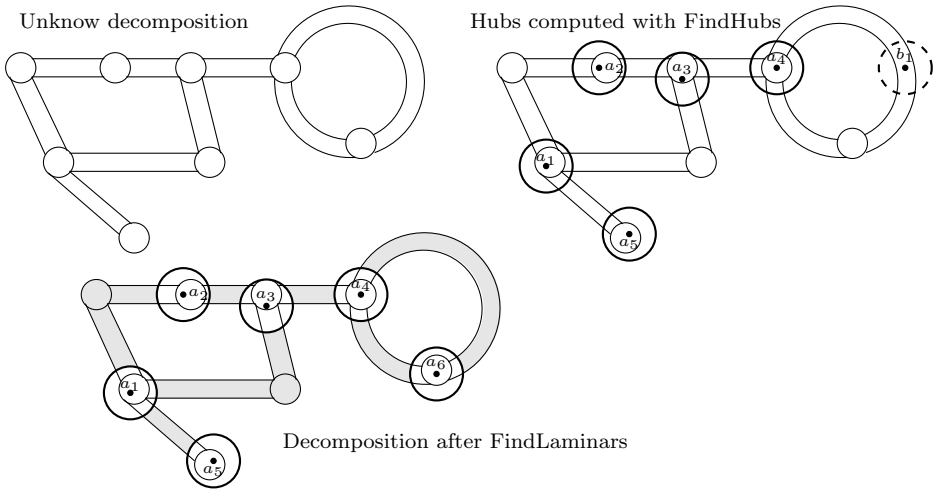


Figure 2: Illustration of the different steps of the algorithm.

The underlying idea of the algorithm behind Theorem 3 is to use BFS (Breadth-first search) to compute shortest paths and their K-neighborhoods, K being chosen large enough to dominate every laminar traversed by the considered shortest paths, but small enough compared to ℓ to detect all hubs of degree at least 3.

The first step, yields two sets of hub-centers A and B, respectively called *unmovable* and *movable* hub-centers, which will be used to determine the laminars. An unmovable hub center $a \in A$ corresponds to exactly one hub center $h \in H$ and is such that $d(a, h)$ is bounded. We show that A contains exactly one such vertex for every hub-center of H whose degree is not 2.

The laminars are determined in a second step which links the hub-centers of the previous step by shortest paths.

Acknowledgments

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