On directed versions of the 1-2-3 Conjecture and the 1-2 Conjecture

Mariusz Woźniak — AGH University, Krakow, Poland

Abstract

Let $G = (V, E)$ be a graph. Given an integer k, a k-coloring (labeling) of G is a function $f : E \to \{1, 2, \ldots, k\}$. The coloring f can be represented by substituting each edge e of G by a multiedge with multiplicity $f(e)$. The degree of x in the respective multigraph equals the sum of labels around a vertex x. The 1-2-3 Conjecture says that for graphs without isolated edges there exists a 3-coloring f such the the corresponding multigraph is locally irregular *i.e.* for each edge xy of G we have $\sigma(x) \neq \sigma(y)$ where $\sigma(x) = \sum_{e \ni x} f(e)$. The 1-2 Conjecture refers to the case when we also color the vertices.

During the talk we shall look at the directed versions of these problems.

1 Introduction

The origins of the problem go back to the eighties of the twentieth century and are associated with attempts to define the notion of irregularity of a graph using labels (colors) on the edges of a graph.

Let $G = (V, E)$ be a graph. Given an integer k, a k-edge-coloring (labeling) of G is a function $f : E \to \{1, 2, \ldots, k\}$. For $x \in V$, we put $\sigma(x) = \sum_{e \ni x} f(e)$. We say that two vertices x, y are *distinguished* if $\sigma(x) \neq \sigma(y)$. The *local irregularity strength* of G is the minimum k such that there exists a k-coloring f distinguishing adjacent vertices in the graph G. The coloring f can be represented by substituting each edge e by a multiedge with multiplicity $f(e)$. The sum $\sigma(x)$ of labels around a vertex x is then equal to the degree of x in the respective multigraph.

Such a local variant of the *irregularity strength* (where we distinguish all vertices) gained great popularity in the twenty first century due to the following nice conjecture of Karoński, Łuczak, and Thomason [7], called the **1-2-3 Conjecture** nowadays (see, for instance the survey paper by Seamone [9]).

Conjecture 1. *Let $G = (V, E)$ be a graph such that G contains no isolated edge K_2 (as a component). Then, there is a coloring $f : E \to \{1, 2, 3\}$ of G such that $\sigma(x) \neq \sigma(y)$ whenever vertices x, y are adjacent in G.*

Note that for a given k-coloring of a graph G, if adjacent vertices are distinguished by σ, then σ defines a proper coloring of the vertices of G. For this reason, the smallest k in a neighbor distinguishing k-coloring f of G is denoted by $\chi_\sigma(G)$.

So, the 1-2-3 Conjecture states that $\chi_\sigma(G) \leq 3$ for every graph G without isolated edges.

The conjecture is still open. The best currently known general upper bound stating that $\chi_\sigma(G) \leq 5$ is due to Kalkowski, Karoński and Pfender [6]. The conjecture is also known to hold for particular graph classes, *e.g.*, bipartite graphs. In this case we have an easy characterization of the family of graphs with $\chi_\sigma(G) = 3$ (see the paper of Thomassen, Wu and Zhang [11]).

If f is a total k-coloring *i.e.* is a function $f : V \cup E \to \{1, 2, \ldots, k\}$ and for $x \in V$, we put $\sigma^t(x) = f(x) + \sum_{e \ni x} f(e)$, we get the following conjecture called for obvious reasons 1-2 **Conjecture**. This conjecture was stated by Przybyło and Woźniak in [10].

Conjecture 2. *Let $G = (V, E)$ be a graph.Then, then there is a coloring $f : V \cup E \to \{1, 2\}$ of G. such that $\sigma^t(x) \neq \sigma^t(y)$ whenever vertices x, y are adjacent in G.*

2 Digraphs and 1-2-3 Conjecture

If we are dealing with a digraph $D = (V, A)$ and an arc coloring $f : A \to \{1, 2, \ldots, k\}$ then at a vertex $x \in V$ we have out-going arcs $xy \in A$ and in-coming arcs $yx \in A$; accordingly, the vertex x can be sum-characterized either by the *out-sum* $\sigma^+(x) = \sum_{xy \in A} f(xy)$ or by the *in-sum* $\sigma^-(x) = \sum_{yx \in A} f(yx)$ (both corresponding to f) and both sums can be used to distinguish vertices connected with an arc.

Thus, there are four natural possibilities of the above distinguishing in digraphs and all of these variants have already been considered. The first problem of this type was introduced by Borowiecki, Grytczuk, and Pilśniak, [4], and concerned so-called *relative sums*, defined for a vertex x as $\sigma_\pm(x) = \sigma^+(x) - \sigma^-(x)$. The least k so that a k-coloring of a given digraph $D = (V, A)$ exists with $\sigma_\pm(x) \neq \sigma_\pm(y)$ for every arc $xy \in A$ is denoted by $\overrightarrow{\chi}_\pm(D)$. The authors proved in [4] that the following theorem holds.

Theorem 3. $\overrightarrow{\chi}_\pm(D) \leq 2$ *for any digraph* D. ■

Baudon, Bensmail, and Sopena considered the least integer k admitting a k-coloring of a digraph $D = (V, A)$ such that $\sigma^+(x) \neq \sigma^+(y)$ for every $xy \in A$. We denote such k by $\overrightarrow{\chi}_+(D)$. In [3] the authors showed that

Theorem 4. $\overrightarrow{\chi}_+(D) \leq 3$ *for every digraph* D

(The requirement $\sigma^-(x) \neq \sigma^-(y)$ for $xy \in A$ leads to the parameter $\overrightarrow{\chi}_-(D)$, which is closely related to $\overrightarrow{\chi}_+(D)$, since $\overrightarrow{\chi}_-(D) = \overrightarrow{\chi}_+(D^{-1})$, where D^{-1} is created from D by reversing the direction of each arc in D.

The third natural variant was suggested by Łuczak [8], who proposed to study the distinguishing requirement $\sigma^+(x) \neq \sigma^-(y)$ for $xy \in A$. Barme *et al.* [1] observed that the corresponding parameter $\overrightarrow{\chi}_L(D)$ is not defined provided that D has an arc xy satisfying $d^+(x) = 1 = d^-(y)$, called a *lonely arc*. Indeed, in such a case $\sigma^+(x) = f(xy) = \sigma^-(y)$, and so, regardless from f, x cannot be distinguished from y in an appropriate way. Nevertheless, the following upper bound holds ([1]).

Theorem 5. *If* D *is a digraph without lonely arcs, then* $\overrightarrow{\chi}_L(D) \leq 3$. ■

The inverse (in a way) of the problem of Łuczak above, requiring that $\sigma^-(x) \neq \sigma^+(y)$ for $xy \in A$ (which seems to be the last natural issue in this field) was considered by Horňák, Przybyło, and Woźniak in [5]. We denote the corresponding graph invariant by $\overleftarrow{\chi}_L(D)$.

At first glance, the problems of determining $\overrightarrow{\chi}_L(D)$ and $\overleftarrow{\chi}_L(D)$ seem to be very similar to each other, but in fact the differences between them are rather big. First of all, families of graphs for which the parameter does not exist are different in both cases. Consider a k-coloring f of a digraph $D = (V, A)$. For a vertex $x \in V$, we denote by $A^-(x)$ ($A^+(x)$) the set of arcs in D in-coming to x (out-going from x, respectively). An arc $xy \in A$ is called a *source-sink arc*, an *s-s arc* for short, if x is a source and y is a sink of D (*i.e.*, $d^-(x) = 0$ and $d^+(y) = 0$). Then, inevitably, the sets $A^-(x)$ and $A^+(y)$ are empty, so $\sigma^-(x) = 0 = \sigma^+(y)$ and it is impossible to distinguish the vertices x and y according to the rule above. The situation is similar if both arcs xy and yx belong to A and xy is an s-s arc in the digraph $D' = D - yx$. We then say that $\{xy, yx\}$ is a *source-sink edge* (an *s-s edge* for short). Then, although $A^-(x)$ and $A^+(y)$ are not empty, they are equal to each other, namely, we have $A^-(x) = A^+(y) = \{yx\}$, and hence $\sigma^-(x) = f(yx) = \sigma^+(y)$.

It is easy to see that if we forbid these two configurations in D, then $A^-(x) \neq A^+(y)$ for every arc $xy \in A$, and thus there exists a k-coloring of D with $\sigma^-(x) \neq \sigma^+(y)$ for every $xy \in A$ for sufficiently large k.

The following fact is somewhat surprising.

Proposition 6. *For any integer $k \geq 2$ there is a digraph D_k with $\overleftarrow{\chi}_L(D_k) \geq k$.*

So, the parameter $\overleftarrow{\chi}_L$ is not bounded from above by an absolute constant. Since all the digraphs for which the parameter $\overleftarrow{\chi}_L$ is large contain lonely arcs, the natural question is whether for digraphs without such arcs, three colors would not be enough. The question remains as yet unanswered. A positive answer to this question was conjectured in [5], where the following theorem was proved.

Theorem 7. *If D is a digraph without s-s arcs and without lonely arcs, then $\overleftarrow{\chi}_L(D) \leq 4$.*

Note that forbidding lonely arcs in a digraph D forbids s-s edges in D, too.

3 Digraphs and 1-2 Conjecture

Until now, when it comes to total coloring of digraphs and appropriate variants of distinguishing neighboring vertices, only the case of out-sums and a natural directed variant of the 1-2 Conjecture was considered. So, assume f is a total coloring of a digraph D. To every vertex x, we associate the sum $\sigma^t_+(x)$, where

$$\sigma^t_+(x) := f(x) + \sigma^+(x) = f(x) + \sum_{xy \in A} f(xy).$$

We say that f distinguishes the neighbors if $\sigma^t_+(x) \neq \sigma^t_+(y)$ for each arc $xy \in A$. Again, the least number $k \geq 1$ of colors needed to distinguish adjacent vertices by a k-total coloring of D is denoted by $\overrightarrow{\chi}^t_+(D)$.

Due to Theorem 4, clearly we have $\overrightarrow{\chi}^t_+(D) \leq 3$ for every digraph D (start from an arc coloring and put $f(x) = 1$ for all vertices). As a straight directed analogue of the 1-2 Conjecture, one could naturally wonder about the following question.

Question For every digraph D, do we have $\overrightarrow{\chi}^t_+(D) \leq 2$?

Unfortunately, easy counterexamples to this question can be exhibited, showing that 3 is actually the best general upper bound on $\overrightarrow{\chi}^t_+(D)$.

So, the directed analogue of the 1-2 Conjecture is false. However, as it turns out, one can modify slightly the distinguishing expression so that the corresponding directed analogue of the 1-2 Conjecture is true. It is enough if we consider ordered pairs $(f(x), \sigma^+(x))$ instead of sums $f(x) + \sigma^+(x)$. The following theorem was proved in [2].

Theorem 8. *For every digraph D, there is a total coloring $f : V \cup A \to \{1, 2\}$ such that for each arc $xy \in A$ we have $(f(x), \sigma^+(x)) \neq (f(y), \sigma^+(y))$.*

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