On minimal triangle-free 6-chromatic graphs

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Abstract

A graph with chromatic number k is called *k-chromatic*. Using computational methods, we show that the smallest triangle-free 6-chromatic graphs have at least 32 and at most 40 vertices.

We also determine the complete set of all triangle-free 5-chromatic graphs up to 24 vertices. This implies that Reed's conjecture holds for triangle-free graphs up to at least 24 vertices. Next to that, we determine that the smallest regular triangle-free 5-chromatic graphs have 24 vertices. Finally, we show that the smallest 5-chromatic graphs of girth at least 5 have at least 29 vertices and that the smallest 4-chromatic graphs of girth at least 6 have at least 25 vertices.

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colours required to colour the vertices of G such that no two adjacent vertices have the same colour. A graph G with $\chi(G) = k$ is called *k-chromatic*. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively, or just δ and Δ if G is clear from the context. The *girth* of a graph is the length of its shortest cycle. A triangle-free k-chromatic graph of order n is called a (k, n)-graph, a (k, n, d)-graph a (k, n)-graph with $\Delta = d$ and a $(k, n, \leq d)$-graph a (k, n)-graph with $\Delta \leq d$. Finally, $n(k)$ is defined as the number of vertices of the smallest triangle-free k-chromatic graph.

In [10] Mycielski presented a construction which shows that there are triangle-free graphs with arbitrarily large chromatic number, i.e. $n(k)$ is well-defined. When the Mycielski construction is applied to a triangle-free k-chromatic graph on n vertices, it yields a triangle-free $(k+1)$-chromatic graph on $2n + 1$ vertices, hence $n(k + 1) \leq 2n(k) + 1$.

Chvátal [3] showed that the Grötzsch graph (see Figure 1) – which is obtained by applying the Mycielski construction to the 5-cycle – is the smallest triangle-free 4-chromatic graph, so $n(4) = 11$.

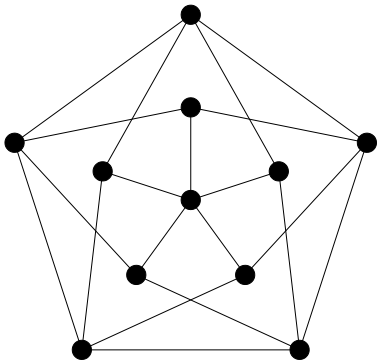


Figure 1: The Grötzsch graph.

In [13] Toft asked for the value of $n(5)$. Using a computer search Grinstead, Katinsky and Van Stone [7] showed that $21 \leq n(5) \leq 22$. In [8], also using a computer search, Jensen and Royle showed that $n(5) = 22$ and that there are at least 80 $(5, 22)$-graphs.

Later, Jensen and Toft asked [9] for the value of $n(6)$. Note that applying the Mycielski construction to the smallest triangle-free 5-chromatic graphs found by Jensen and Royle gives an upper bound of $n(6) \leq 45$. This was the best known upper bound until Droogendijk [4] presented

a $(6, 44)$-graph in 2015. Later he reported that he also found such graphs on 43 vertices, but did not provide any adjacency lists or details. The fact that $n(5) = 22$ and some maximum degree properties imply that $29 \leq n(6)$.

We will present two methods for constructing triangle-free k-chromatic graphs, that is: the maximum triangle-free method and the maximum degree extension method. The former method constructs *maximal* triangle-free graphs, i.e.: triangle-free graphs such that the insertion of any new edge forms a triangle. The principal algorithm of the maximum degree extension method mainly builds upon the following proposition.

Proposition 1. *Let G be an mtf (k, n, d)-graph and let v be a vertex of maximum degree. Then $H = G \setminus (N(v) \cup \{v\})$ is a $(\alpha, n - d - 1, \leq d - 1)$-graph (with $\alpha \in \{k - 1, k\}$).*

Using a combination of these methods allowed us to improve the lower bound for $n(6)$ to 32. We also show that if a $(6, 32)$-graph exists, it must have maximum degree 6, and that if a $(6, 33)$-graph exists, it must have maximum degree 6 or 7.

We also determine the complete set of all $(5, n)$-graphs up to $n = 24$. These counts are listed in Table 1. These graphs can also be downloaded from the *House of Graphs* [1] at `http://hog.grinvin.org/TrianglefreeKChrom`. This implies that Reed's ω, Δ, and χ conjecture [11] holds for triangle-free graphs up to at least 24 vertices. Next to that, we determine the smallest regular triangle-free 5-chromatic graphs. That is:

Theorem 2. *The smallest regular triangle-free 5-chromatic graphs have 24 vertices. There are exactly 63 such graphs and all of them are 7-regular.*

n	Maximum degree							
	< 7	7	8	9	10	11	12	> 12
22	0	8	70	2	0	0	0	0
23	0	16 033	257 922	41 067	434	1	0	0
24	0	3 735 593	687 757 507	327 307 106	11 219 245	58 283	28	0

Table 1: The counts of the smallest $(5, n)$-graphs according to their maximum degree.

In [9] Jensen and Toft also asked for the order of the smallest 4-chromatic graph of girth at least 5. The Brinkmann graph, which is the smallest 4-regular 4-chromatic graph of girth 5 [2], gives an upper bound of 21 vertices. Recently, Royle [12] showed that the smallest 4-chromatic graphs of girth at least 5 have 21 vertices and that there are exactly 18 such graphs. Next to the Brinkmann graph, this set also contains the smallest 4-chromatic P_{12}-free graph of girth 5 found by Schaudt and the author [6].

We determine the complete set of all 4-chromatic graphs of girth 5 on 22 vertices. Figure 2 shows the smallest 4-critical graph of girth 5 and maximum degree 7 – it has 22 vertices. (A *k-critical* graph is a k-chromatic graph for which every proper subgraph is $(k - 1)$-colourable).

By adapting the maximum degree extension technique for graphs of higher girth, we also show the following.

Theorem 3. *The smallest 5-chromatic graph of girth at least 5 has at least 29 vertices.*

Theorem 4. *The smallest 4-chromatic graph of girth at least 6 has at least 25 vertices.*

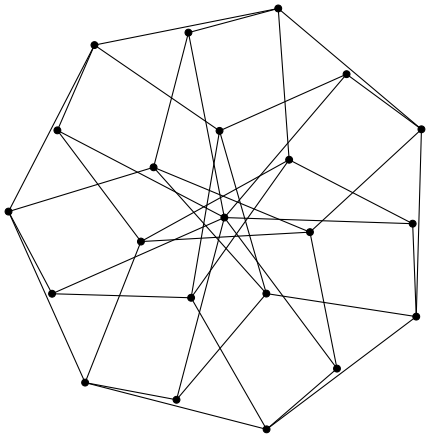


Figure 2: The smallest 4-critical graph of girth 5 and maximum degree 7.

We will also describe how, using a heuristic method, we constructed more than 650 000 $(6, 40)$-graphs which improves the upper bound for $n(6)$ from 44 to 40.

To summarise, our main result is:

Theorem 5. $32 \leq n(6) \leq 40$.

Applying the Mycielski construction to a $(6, 40)$-graph yields a $(7, 81)$-graph and by applying the maximum degree extension technique using the lower bound from Theorem 5 we obtain:

Corollary 6. $40 \leq n(7) \leq 81$.

More information can be found on the arXiv in [5].

References

[1] G. Brinkmann, K. Coolsaet, J. Goedgebeur, and H. Mélot. House of Graphs: a database of interesting graphs. *Discrete Applied Mathematics*, 161(1-2):311–314, 2013. Available at `http://hog.grinvin.org/`.

[2] G. Brinkmann and M. Meringer. The smallest 4-regular 4-chromatic graphs with girth 5. *Graph Theory Notes of New York*, 32:40–41, 1997.

[3] V. Chvátal. The minimality of the Mycielski graph. In *Graphs and combinatorics*, pages 243–246. Springer, 1974.

[4] L. Droogendijk. A triangle-free 6-chromatic graph with 44 vertices. `https://math.stackexchange.com/questions/1561029/a-triangle-free-6-chromatic-graph-with-44-vertices`, 2015.

[5] J. Goedgebeur. On minimal triangle-free 6-chromatic graphs. *arXiv preprint arXiv:1707.07581*, 2017.

[6] J. Goedgebeur and O. Schaudt. Exhaustive generation of k-critical $\mathcal{H}$-free graphs. *Journal of Graph Theory*, 87(2):188–207, 2018.

[7] C.M. Grinstead, M. Katinsky, and D. Van Stone. On minimal triangle-free 5-chromatic graphs. *Journal Of Combinatorial Mathematics And Combinatorial Computing*, 6:189, 1989.

[8] T. Jensen and G.F. Royle. Small graphs with chromatic number 5: a computer search. *Journal of Graph Theory*, 19(1):107–116, 1995.

[9] T. Jensen and B. Toft. *Graph coloring problems*. A Wiley Interscience Publication. John Wiley & Sons, 1995.

[10] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.

[11] B. Reed. ω, Δ, and χ. *Journal of Graph Theory*, 27(4):177–212, 1998.

[12] G.F. Royle. The smallest 4-chromatic graphs of girth 5. `https://mathoverflow.net/questions/193716/what-is-the-smallest-4-chromatic-graph-of-girth-5`, 2015.

[13] B. Toft. 75 graph-colouring problems, 1988. Privately circulated booklet.