An automaton describing left burnable configurations in the sandpile model on the ladder graph $\mathbf{Z} \times P_H$

Henri Derycke — **Univ. Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France**

Abstract

The sandpile model is well defined for any finite graph. Járai and Lyons approximated the behaviour on the bi-infinite ladder of height H in $\mathbb{Z}^2$ by a series of rectangles of height H and increasing width. This leads them in particular to the concept of left burnable configurations as a subset of recurrent configurations. We are interested in automata recognizing column by column these configurations. Járai and Lyons give an explicit automaton for the height $H = 2$ and have shown that there is an automaton for all H with rough bound $(2^H)^{2^H}$ over the number of states. Based on a bijection between the recurrent configurations and the spanning trees and on a combinatorial study, I propose a rough bound on the number of states in $\alpha^{H \log H}$. The algorithms linked to this bound make it possible to produce the automata for the height $H = 3$ with 13 states and $H = 4$ with 76 states. In this construction, the automaton contains at least β^H states because a subset of these states is in bijection with separable permutations (those avoiding 3142 and 2413).

The Abelian sandpile model was introduced by physicists Bak, Tang and Wiesenfeld in [1] as a model of self-organized criticality. Given a simple, undirected graph $(V \cup \{s\}, E)$ where we distinguish s as the sink of the graph, we consider *configurations* in this model which are an assignments $\eta : V \mapsto \mathbf{N}$ of some grains of sand on each vertex. We say that η is stable at $x \in V$ if $\eta(x) < \deg(x)$, and η is stable if it is stable at all $x \in V$. If η is unstable at x, then x is allowed to topple which means that x sends one grain along each edge incident to it. Grains arriving at the sink are lost. Given a configuration η, we define a stabilization as a sequence of allowed toppling until a stable configuration is reached. The result of all stabilizations is unique due to commutation of toppling of unstable vertices and is noted $stab(\eta)$.

Let $P(\eta)$ be the result of a stabilization of $\eta + \mathbf{1}_{s\sim}$, that is η with an extra grain on each neighbour of s, which may be interpreted as a forced toppling of the sink. The set of recurrent configurations is a subset of the stable configurations characterized by Dhar [5] as the fixed points of P. For such a fixed point, each vertex topples exactly once in this process.

The notion of recurrence is related to a natural Markov chain in this model not discussed here [4], and it is well studied for its connection with spanning trees [5], uniform spanning tree, the Tutte polynomial on the underlying graph [3, 8].

We are interested in the recurrent configuration on a family of ladder graph of height $H \geq 2$ defined as follow. Let $\Lambda_{W,H}$ be the finite graph derived from the ladder graph $P_W \times P_H$, rooted on the left side to another vertex and with extra vertices on each line on the right side (Figure 1). A stable configuration on $\Lambda_{W,H}$ is a word on the alphabet $\{0,1,2\} \times \{0,1,2,3\}^{H-2} \times \{0,1,2\}$ listing the number of grains on a column's vertices.

In this paper, we present the construction of an automaton that accepts all the recurrent configurations of the graphs $\Lambda_{\bullet,H} = \{\Lambda_{W,H} \mid W > 0\}$. The automaton will respect the same properties than the description given by Járai and Lyons in [7] while improving the rough bound on states from $(2^H)^{2^H}$ to $2^H H!$. Gamlin suggests the existence of an automaton with at most α^H states [6, Appendix] as for spanning trees which can be read column by column from left to right by recording in a non-crossing partition the connected components at left of the current column. An explanation for not reaching Gamlin's conjecture is that we use a bijection between spanning

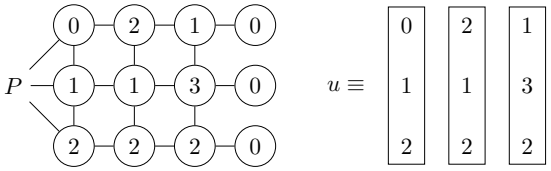


Figure 1: A recurrent configuration on $\Lambda_{3,3}$

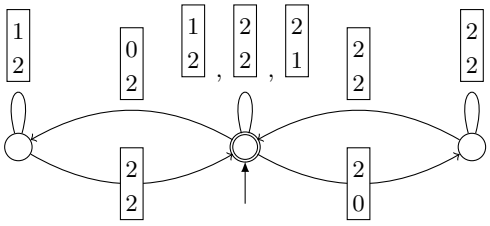


Figure 2: Automaton for $H = 2$

trees and recurrent configurations which requires to rank some knowledge along each cycle induced by each external edge added to the spanning tree which may extends far also *at right* of the current column. We succeed in limiting this knowledge at right to a kind of permutation apparently related to the $H!$ factor in our upper bound, but by our approach it seems difficult to remove it. This construction is based on *decreasing edge-vertex traversal* introduced by Cori and Le Borgne [3], an object encoding both the spanning tree and the recurrent configuration in bijection.

Let $<_e$ be an order on E. An *edge-vertex traversal* $\sigma = (\sigma_i)_{1 \leq i \leq |V|+1+|E|}$ of $G = (V \cup \{s\}, E)$ is a permutation over the $V \cup \{s\} \cup E$. We note $\sigma^{<i} = \{\sigma_j \mid j < i\}$. A *decreasing edge-vertex traversal* is an edge-vertex traversals such that (i) if σ_i is a vertex of V, then σ_{i-1} is an edge incident to σ_i, (ii) if σ_i is an edge, it is minimal with respect to $<_e$ among all the edges not in $\sigma^{<i}$ and incident to a vertex of $\sigma^{<i}$.

Lemma 1 (Cori,Le Borgne[3])**.** *The recurrent configurations and the decreasing edge-vertex traversals are in bijection.*

The bijection can be seen as the process of the fixed point characterization by Dhar, treating each grain one after the other following the order over the edges. A corollary is that the number of grains of a vertex x is given by the number of edges incident to x that appear after x in σ. And, the spanning tree is deduced from the set of edges that precede a vertex in the traversal. Figure 3 (a) illustrates an example of decreasing traversal. Each vertex and each edge (in green) is labelled by its position in σ.

Let $<_e$ be the geometric order on the edges of $\Lambda_{\bullet,H}$ from left to right, ties broken from top to bottom. We note D_H the set of decreasing traversals on the $\Lambda_{\bullet,H}$. Let $\sigma \in D_H$ a decreasing traversal on $\Lambda_{W,H}$, we decompose σ in a sequence $(c_i)_{1 \leq i \leq W}$ of *column decreasing traversals* where c_i is the subsequence of σ on the vertices of the i-th column of $\Lambda_{W,H}$ and the edges incident to them (Figure 3 (b)).

Property 1. *The set of decreasing traversals and their decomposition in column decreasing traversals are in bijection.*

Proof. Let $(c_i)_{1 \leq i \leq W}$ be a decomposition. Each c_i contains the order of the vertices of the $i-th$ column and its incident edges. Thus from the corollary, we have the number of grains in each vertices. The bijection of Cori-Le Borgne provides the original decreasing traversal. □

Let C_H be the set of the column decreasing traversals that appears in the decomposition of some $\sigma \in D_H$.

Theorem 1. *A $(c_i)_{1 \leq i \leq W} \in (C_H)^W$ is the decomposition of decreasing traversal if and only if:*

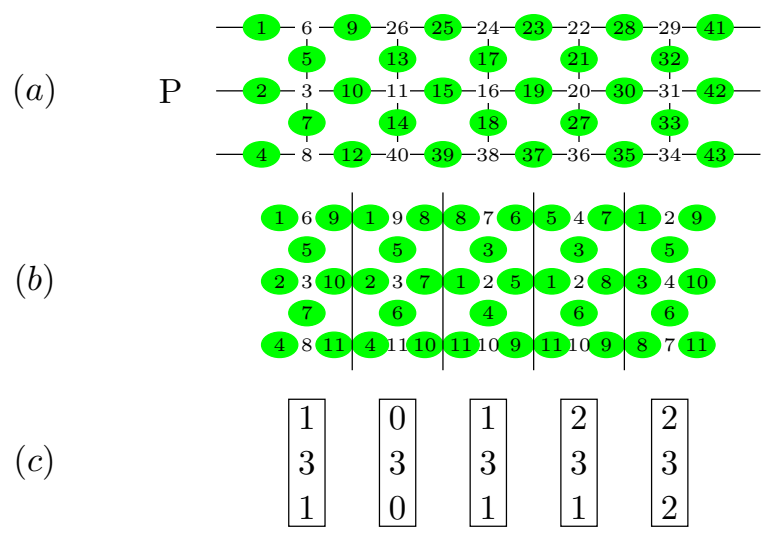


Figure 3: Column decreasing decomposition and configuration

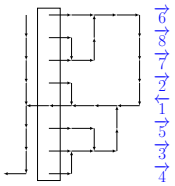


Figure 4: Tree and separable permutation

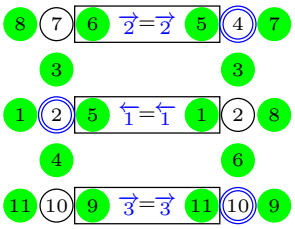


Figure 5: Compatibility rule

1. *the first H edges in c_1 are the H left horizontal edges from top to bottom.*
2. *the last H elements of c_W are the H right horizontal edges from top to bottom.*
3. *for all $1 \leq i < W$, the order in c_i et c_{i+1} of the H horizontal edges incident both to columns i and $i+1$ respect:*
   (a) *the orders in c_i and c_{i+1} are the same*
   (b) *for each edge, if it appears before its end in c_i, it appears after its end in c_{i+1} and vice versa.*

The item 3. of Theorem 1 is compatibility rule illustrated by Figure 5. Let $\sigma \in D_H$. For each horizontal edge $e = x \sim y$, either $\sigma = \dots x \dots e \dots y \dots$ or $\sigma = \dots y \dots e \dots x \dots$. The first endpoint is marked with a double circle. The blue numbers with arrows hat are the information that describes compatibility. Given $c \in C_H$, we can derive two vectors $left(c)$ and $right(c)$ describing for each side the order of its horizontal edges and there orientation toward their first endpoint.

Sketch of the proof. First, we show that these are necessary conditions. The condition 1 and 2 are direct consequences of the construction of [3]. The edges incident to the sink are the first in a decreasing traversal and the last edges are the edges incident to the extra vertices on the right. Condition 3a is straightforward. From [3] we have that an edge appears after its endpoints if and only if it is externally active in the tree, that is, it is maximal according to $<_e$ in its induced cycle. One can show that the carefully chosen order $<_e$ guarantees that the maximal edge of any cycle is vertical. Then, an horizontal edge is not externally active so, it is not after both endpoints.

The proof that the conditions are sufficient is based on the completion of an linear extension step by step looking between two consecutive columns. The horizontal edges split the graph in two disjoint part. Then the order on the left is not relevant for the order on the right as long as the order of the shared edges is the same. □

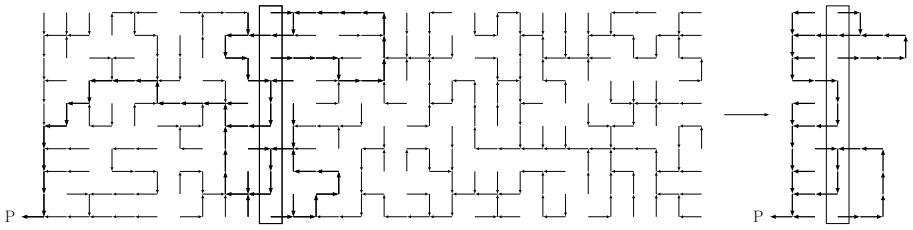


Figure 6: Simplification from a generic tree

Theorem 1 provides the description of an automaton with alphabet C_H. For each $c \in C_H$, we have exactly one transition from $right(c)$ to $left(c)$. Then the states of the automaton are the set of $left(c)$ and $right(c)$ for $c \in C_H$.

The number of states is bound by $2^H H!$ and the number of transitions is $|C_H|$. The automaton is derived from C_H. However we do not have a combinatorial description of D_H yet, but the following property limits the enumeration to graphs $\Lambda_{W,H}$ with width W lower than $2H + 3$.

Property 2. *The set C_H is also the set of the column decreasing traversal that appears in the decomposition of the $\sigma \in D_H$ of $\Lambda_{2H+3,H}$.*

Sketch of the proof. Given a $c \in C_H$ and a tree T that produces c, it's possible to apply transformations on T that preserve c (Figure 6). The tree T is pruned from the branches that don't lead to a vertex of c or a adjacent vertex of c. Then the branch on the left can be reduced to an imbrication of combs. The transformation on the right has to preserve the order of the branches according to their maximal edges. The resulting tree needs at most H columns on the left and H columns on the right. □

As a by product, this construction gives an injection from separable permutations [2] to a subset of states ans a exponential lower bound on the states. A separable permutation can be defined by a binary tree that can be embedded in $\Lambda_{\bullet,H}$ as in Figure 4.

From this automaton, we have an automaton on the configurations by projection. However, this projection loses the lower and upper bound.[1]

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[1] see `https://www.labri.fr/perso/hderycke/ladder_automata.html` for the automata for height 3 and 4