Optimal General Matchings

Szymon Dudycz — **University of Wroclaw**
Katarzyna Paluch — **University of Wroclaw**

Abstract

Given a graph $G = (V, E)$ and for each vertex v a subset $B(v)$ of the set $\{0, 1, \ldots, d_G(v)\}$, where $d_G(v)$ denotes the degree of vertex v, a B-matching of G is any set $F \subseteq E$ such that $d_F(v) \in B(v)$ for each vertex v. The general matching problem asks the existence of a B-matching in a given graph. A set $B(v)$ is said to have a gap of length p if there exists a number $k \in B(v)$ such that $k+1, \ldots, k+p \notin B(v)$ and $k+p+1 \in B(v)$. Without any restrictions the general matching problem is NP-complete. However, if no set $B(v)$ contains a gap greater than 1, then the problem can be solved in polynomial time[5]. In this paper we consider a problem of finding maximum cardinality general matching.

We present the first polynomial time algorithm for the maximum B-matching for the case when no set $B(v)$ contains a gap greater than 1. This also yields the first pseudopolynomial algorithm for the weighted version of the problem.

Given a graph $G = (V, E)$ and for each vertex $v \in V$ a subset $B(v)$ of the set $\{0, 1, \ldots, d_G(v)\}$, where $d_G(v)$ denotes the degree of vertex v in the graph G, a B-matching of G is any set $F \subseteq E$ such that $d_F(v) \in B(v)$ for each vertex v, where $d_F(v)$ denotes the number of edges of F incident to v. The general matching problem asks the existence of a B-matching in a given graph. A set $B(v)$ is said to have a *gap of length* p if there exists a natural number $k \in B(v)$ such that $k+1, \ldots, k+p \notin B(v)$ and $k+p+1 \in B(v)$. Without any restrictions the general matching problem is NP-complete [13]. However, for the case when no set $B(v)$ contains a gap of length greater than 1, Lovasz [13] developed a structural description and Cornuejols [5] presented a polynomial time algorithm for finding a B-matching, if it exists. It is then one of the strongest generalizations of matchings which is polynomially solvable, unless $\mathcal{P} = \mathcal{NP}$. In the maximum/minimum cardinality variant the goal is to find a B-matching having a maximum/minimum number of edges. In the weighted version of the problem a weight function $w : E \to \mathbb{R}$ is given and the aim is to find a B-matching that maximizes or minimizes the sum of the weights of the edges.

Previous work If $B(v) = \{0, 1\}$ for each vertex v, then a B-matching is in fact a *matching*, i.e., a set of vertex-disjoint edges. A *perfect matching* is a B-matching such that $B(v) = 1$ for each vertex v. Given a function $b : V \to \mathbb{N}$, a b-matching is any set $F \subseteq E$ such that $d_F(v) \leq b(v)$ for each vertex v and a perfect b-matching or a b-factor is any set $F \subseteq E$ such that $d_F(v) = b(v)$ for each vertex v. If in addition to a function b we are also given a function $a : V \to \mathbb{N}$, then an (a, b)-matching is any set $F \subseteq E$ such that $a(v) \leq d_F(v) \leq b(v)$ for each vertex v.

All these special cases of the general matching problem are well-solved, both in unweighted and weighted versions. For instance, for the maximum weight b-matching there exist algorithms with the following running times: $\mathcal{O}(n^2 B)$ by Pulleyblank [18], $\mathcal{O}(n^2 m \log B)$ by Marsh [15], $\mathcal{O}(m^2 \log n \log B)$ by Gabow [6], $\mathcal{O}(n^2 m + n \log B(m + n \log n))$ and $\mathcal{O}(n^2 \log n(m + n \log n))$ by Anstee [1], and $\tilde{\mathcal{O}}(W\phi^\omega)$ by Gabow and Sankowski [7], where $n = |V|$, $m = |E|$, $B = \max b(v)$, $\phi = \sum b(v)$ and n^ω is the time required to multiply two $n \times n$ matrices. For a good survey on these problems see [21].

In the *antifactor* problem for each vertex v we have $|\{0, 1, \ldots, d_G(v)\} \setminus B(v)| = 1$, meaning that for each vertex there is exactly one degree excluded from the set $B(v)$. Graphs that have an antifactor have been characterized by Lovasz in [12].

For the more general case when no set $B(v)$ contains a gap of length greater than 1, Cornuejols [5] in 1988 presented two solutions to the problem of finding such B-matching, if it exists. One uses a reduction to the edge-and-triangle partitioning problem, in which we are given a graph $G = (V, E)$ and a set T of triangles (cycles of length 3) of G and are to decide if the set of vertices V can be partitioned into sets of cardinality of 2 and 3 so that each set of cardinality 2 is an edge of E and each set of cardinality 3 is a triangle of T. The other is based on an augmenting path approach applied in the modified graph $G' = (V \cup V', E')$ in which each edge e of G is split with two new vertices into three edges. For each new vertex v' the set $B(v')$ is defined to be $\{1\}$ and we start from the set $F \subseteq E'$ such that all requirements regarding vertices of G are satisfied, i.e., $d_F(v) \in B(v)$ for each vertex $v \in V$ and for each vertex $v' \in V'$ it is $d_F(v') \leq 1$. Next we aim to gradually augment F so that it also satisfies the requirements regarding new vertices V' and $d_F(v') = 1$ for each $v' \in V'$. In either case, the computed B-matching is not guaranteed to be of maximum or minimum cardinality. A good characterization of graphs that have a B-matching [22] was provided in 1993 by Sebő [22].

General matchings in bipartite graphs were also studied in terms of their parametrized complexity. Gutin et al. showed that for graphs $G = (U \cup V, E)$, such that $|B(u)| = 1$ for every $u \in U$, there exists a fixed-parameter tractable algorithm parametrized by the size of V [8].

For the optimization variant of the general matching with no gap greater than 1 Carr and Parekh provided a linear relaxation which is $\frac{1}{2}$-integral [4].

A B-matching is said to be *uniform* if each $B(v)$ is either an interval, i.e., has the form $\{a(v), a(v)+1, \ldots, b(v)\}$ for some nonnegative integers $a(v) \leq b(v)$ or an interval intersected with either even or odd numbers, i.e., has the form $\{a(v), a(v)+2, \ldots, b(v)\}$ for two nonnegative integers $a(v) \leq b(v)$ such that $b(v) - a(v)$ is even. A maximum/minimum weight uniform B-matching problem was shown to be solvable in polynomial time by Szabó [24]. In the solution to the weighted uniform B-matching Szabó uses the following result of Pap [17]. Let $\mathcal{F}$ be an arbitrary set of odd length cycles of graph G, where a single vertex is considered a cycle of length 1. A *perfect* $\mathcal{F}$-matching is any set of cycles and edges of G such that each vertex belongs to exactly one edge or cycle from $\mathcal{F}$. Pap gave a polynomial time algorithm which minimizes a linear function over the convex hull of perfect $\mathcal{F}$-matchings.

Our results We give the first polynomial time algorithm for the maximum/minimum B-matching and $B(v)$ for the case when no set contains a gap of length greater than 1. Our solution yields also the first pseudopolynomial algorithm for the maximum/minimum weight B-matching for the case when no set $B(v)$ contains a gap of length greater than 1.

We provide a structural result for both cardinality and weighted variants, which states that given two B-matchings M and N, their symmetric difference $M \oplus N = (M \setminus N) \cup (N \setminus M)$ can be decomposed into a set of *canonical paths*, a notion which we define precisely later and which plays an analogous role as that of an *alternating path* in the context of standard matchings. A path P is alternating with respect to a matching M if its edges alternate between edges of M and edges not belonging to M. Roughly speaking, a canonical path (with respect to a given B-matching M) consists of a meta-path, that is a sequence of alternating paths, and possibly some number of meta-cycles attached to the endpoints of this meta-path. A meta-cycle is a sequence of alternating paths such that the beginning of the first alternating path coincides with the end of the last alternating path in the sequence. After the application of a canonical path $\mathcal{P}$ to a B-matching M we obtain another B-matching $M' = M \oplus \mathcal{P}$ such that only the parities of the degrees in M and M' of the endpoints of $\mathcal{P}$ are different.

Equipped with this structural result we show how finding a maximum/minimum B-matching

can be reduced to a series of computations of a maximum/minimum weight uniform B-matching. In fact we prove that in order to verify if a given B-matching M has maximum/minimum weight it suffices to check if there exists a uniform B-matching of so called *neighbouring type* to M, whose weight is greater/smaller than that of M.

Additionally, we show a very simple reduction of a weighted uniform B-matching to a weighted (a, b)-matching, which yields a more efficient and simpler algorithm than the one by Szabó.

The remaining open problem is whether there exists a polynomial time algorithm for a maximum weight B-matching for the case when no set contains a gap of length greater than 1. It is also possible that our algorithm runs in polynomial time.

Motivation Matchings, b-matchings and factors are basic combinatorial notions that lie at the foundation of combinatorial optimization. The general matching problem restricted to gaps of at most 1 is one of the strongest generalizations of matching, that was not proven $\mathcal{NP}$-hard. As such it is of theoretical importance to find a polynomial time algorithm for a maximum/minimum cardinality/weight B-matching with gaps at most 1 or in the case of a maximum weight B-matching, to prove that it is $\mathcal{NP}$-hard.

As for practical applications, the general matching is related to the extended global cardinality constraint problem (EGCC). Given a set of variables X, a set of values D, a domain for each variable $D(x) \subseteq D$ and a cardinality set $K(d)$ for each $d \in D$, the goal is to find a valuation of variables, such that the number of variables with value d belongs to $K(d)$. Algorithms for general matchings were used to solve some restricted variants of EGCC [8, 20], while the general EGCC is $\mathcal{NP}$-hard. The EGCC problem was used for, among others, staff scheduling in healthcare [3], optical network design [23] or car sequencing [19]. For empirical survey on EGCC see [16].

Related work In the deficiency problems the task consists in finding a matching that is as close as possible to given sets $B(v)$. Hell and Kirkpatrick [9] gave an algorithm for finding a minimum deficiency (a, b)-matching among all $(0, b)$-matchings, where the deficiency is measured as the sum of differences $a(v) - d(v)$ over all vertices whose degree is not between $a(v)$ and $b(v)$. They also proved that for another measure of deficiency, namely number of vertices whose degree is outside $(a(v), b(v))$, the problem is NP-hard.

Another related problem consists in decomposing a graph into (a, b)-matchings - a graph that can be decomposed into (a, b)-matchings is called (a, b)-factorable. In [11] Kano gave a sufficient condition for a graph to be $(2a, 2b)$-factorable. Cai [14] generalized this result to $(2a - 1, 2b)$, $(2a, 2b + 1)$ and $(2a - 1, 2b + 1)$ -factorable graphs. Hilton and Wojciechowski showed another sufficient condition for an $(r, r + 1)$-factorization of graphs [10].

(a, b)-matchings were also studied in the stable framework - Biro et al. proved that checking whether a stable (a, b)-matching exists is NP-hard [2].

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