On density of subgraphs of Cartesian products

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Abstract

In this paper, we introduce a notion of VC-dimension VC-dim(G) of a subgraph G of a Cartesian product, generalizing the classical Vapnik-Chervonenkis dimension of set-families (viewed as subgraphs of hypercubes). We then extend a result of Haussler, Littlestone and Warmuth about the density of subgraphs of hypercubes to subgraphs G of Cartesian products $G_1 \times \cdots \times G_m$ of connected graphs. Namely, we prove that if $G_1, \ldots, G_m$ belong to the class $\mathcal{G}(H)$ of all finite connected graphs not containing a given graph H as a minor, then for any subgraph G of $G_1 \times \cdots \times G_m$ the inequality $\frac{|E(G)|}{|V(G)|} \leq \mu(H)\text{VC-dim}(G)$ holds, where $\mu(H)$ is the density of the graphs from $\mathcal{G}(H)$.

1 Introduction

A folklore result asserts that if $G = (V, E)$ is an induced subgraph of the m-dimensional hypercube $Q_m := K_2 \times \cdots \times K_2$, then $\frac{|E|}{|V|} \leq \log |V|$. In [3, Lemma 2.4], Haussler et al. improve this result by replacing $\log |V|$ by the VC-dimension of G (defined in Section 2). In this paper, we extend the inequality $\frac{|E|}{|V|} \leq \text{VC-dim}(G) =: d$ to subgraphs $G = (V, E)$ of Cartesian products $\Gamma := G_1 \times \cdots \times G_m$ of graphs $G_1, \ldots, G_m$ from the class $\mathcal{G}(H)$ of connected graphs avoiding a fixed graph H as a minor. To do so, we define a notion of VC-dimension of subgraphs G of arbitrary Cartesian products and show that if all the factors of Γ belong to $\mathcal{G}(H)$, then the density of G is at most the VC-dimension VC-dim(G) of G times $\mu(H)$, where $\mu(H)$ is a constant such that any graph not containing H as a minor has density at most $\mu(H)$ (it is well known [2] that if $r := |V(H)|$, then $\mu(H) \leq cr\sqrt{\log r}$ for a universal constant c). Namely, we show the following theorem

Theorem 1. *Let H be a graph and let G be a subgraph of a Cartesian product $\Gamma = G_1 \times \cdots \times G_m$ of connected graphs $G_1, \ldots, G_m$ from $\mathcal{G}(H)$. Then*

$$\frac{|E(G)|}{|V(G)|} \leq \mu(H)\,\text{VC-dim}(G) \leq \mu(H) \log |V(G)|.$$

2 Preliminaries

All graphs $G = (V, E)$ occurring in this note are finite, undirected, and simple. The *density* dens(G) of G will be the maximal ratio $|E(G')|/|V(G')|$ over all its subgraphs G'.

Let $G_1, \ldots, G_m$ be a family of m connected graphs. The *Cartesian product* $\Gamma := \prod_{i=1}^m G_i := G_1 \times \cdots \times G_m$ is a graph defined on the set of all m-tuples $(x_1, \ldots, x_m)$, $x_i \in V(G_i)$, where two vertices $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ are adjacent if and only if there exists an index $1 \leq j \leq m$ such that $x_j y_j \in E(G_j)$ and $x_i = y_i$ for all $i \neq j$. The *m-dimensional hypercube* Q_m is the Cartesian product of m copies of K_2.

A *subproduct* Γ' of a Cartesian product $\Gamma = \prod_{i=1}^m G_i$ is a product such that $\Gamma' = \prod_{j=1}^k G'_{i_j}$, where each G'_{i_j} is a connected *non-trivial* (i.e., with at least one edge) subgraph of G_{i_j} (with

$\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$). If every G_{i_j} consists of a single edge, then Γ' is called a *cube-subproduct*. Given a vertex $v' = (v'_{i_1}, \ldots, v'_{i_k})$ of Γ', we say that a vertex $v = (v_1, \ldots, v_m)$ of Γ is an *extension* of v' if $v_{i_j} = v'_{i_j}$ for all $j \in \{1, \ldots, k\}$. We denote by $F(v')$ the set of all extensions v in Γ of a vertex v' of a subproduct $\prod_{j=1}^{k} G'_{i_j}$ and call $F(v')$ the *fiber* of v' in the product Γ. Let G be a subgraph of a Cartesian product Γ and Γ' be a subproduct of Γ. The *trace* of $V(G)$ on $V(\Gamma')$ consists of all vertices v' of Γ' such that $F(v') \cap V(G) \neq \emptyset$. The *projection* of G on Γ' is the subgraph $\pi_{\Gamma'}(G)$ of Γ' induced by the trace of $V(G)$ on $V(\Gamma')$.

The classical definition of VC-dimension (for set systems) can be stated in a graph-theoretical way as follows. A cube-subproduct Γ' of Q_m is *shattered* by G if $\pi_{\Gamma'}(G) = \Gamma'$. The *VC-dimension* of G is the largest dimension of a cube-subproduct shattered by G. More generally, a subproduct $\Gamma' := \prod_{j=1}^{k} G'_{i_j}$ of a Cartesian product $\Gamma = \prod_{i=1}^{m} G_i$ is *shattered* by G if $\pi_{\Gamma'}(G) = \Gamma'$. The *VC-dimension* VC-dim(G) of G with respect to the Cartesian product Γ is the largest number of non-trivial factors in a subproduct Γ' of Γ shattered by G. Equivalently, VC-dim(G) is the largest dimension of a cube-subproduct of Γ shattered by G.

3 Proof of Theorem 1

We start with a brief description of the proof of the inequality $|E|/|V| \leq d$ provided by Haussler et al. [3] for subgraphs of hypercubes. We adapt this proof to our notations and we generalize it to subgraphs of arbitrary Cartesian products. Let $G = (V, E)$ be a subgraph of a Cartesian product of m copies of K_2 (i.e., of the hypercube Q_m) and set $d :=$ VC-dim(G). Assume that the factors K_2 of Q_m are indexed by $1, \ldots, m$ and consider the ith K_2, $i \in \{1, \ldots, m\}$. This K_2 is an edge whose extremities are denoted by u and v. Remark that every vertex of Q_m has either u or v as its ith coordinate and so do the vertices of G. Denote by G_{uv} the graph obtained by projecting G on the Cartesian product of the $m-1$ factors of Q_m not indexed by i (i.e., by identifying the vertices having u as ith coordinate with those having v). Denote by G^{uv} the graph obtained from G by contracting every edge of G of the form $(x_1, \ldots, u, \ldots, x_m)(x_1, \ldots, v, \ldots, x_m)$. Then VC-dim($G_{uv}$) $\leq d$ and VC-dim(G^{uv}) $\leq d-1$ hold. We now do the following induction hypothesis: $|E(G_{uv})| \leq d|V(G_{uv})|$ and $|E(G^{uv})| \leq (d-1)|V(G^{uv})|$. The proof of the required density inequality follows by induction from the equality $|V| = |V(G_{uv})| + |V(G^{uv})|$ and the inequality $|E| \leq |E(G_{uv})| + |E(G^{uv})| + |V(G^{uv})|$.

We continue by defining the graphs G_{uv} and G^{uv} for subgraphs of arbitrary Cartesian products. In case of the hypercube $\prod_{i=1}^{m} K_2$, the edge uv corresponds to a factor of this product and G_{uv} can be viewed as the image of G in the product of K_2's where the whole factor corresponding to uv was contracted. When the factors are arbitrary graphs, contracting a whole factor of the product would be too rough. So, let u and v be two adjacent vertices of some factor G_i. Let N denote the set of common neighbors of u and v in G_i. Let $\widehat{G}_i$ be the graph obtained from G_i by contracting the edge uv, namely, the graph in which the edge uv is replaced by a vertex w and every edge xu and/or xv of G_i is replaced by a single new edge xw; thus $\widehat{G}_i$ does not contain loops and multiple edges. Let $\widetilde{G}_i$ be the graph which is a star having as the central vertex a vertex $\widetilde{w}$ corresponding to the edge uv and as the set $\widetilde{N}$ of leaves the vertices $\widetilde{x}$ corresponding to vertices x of N (i.e., such that xuv is a triangle of G_i); the edges of $\widetilde{G}_i$ are all pairs of the form $\widetilde{w}\widetilde{x}$.

Let G_{uv} be the subgraph of $\widehat{\Gamma} := G_1 \times \cdots \times G_{i-1} \times \widehat{G}_i \times G_{i+1} \times \cdots \times G_m$ obtained from G by identifying every vertex $(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_m)$ with its neighbor $(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_m)$ and by removing multiple edges. Let G^{uv} be the subgraph of $\widetilde{\Gamma} := G_1 \times \cdots \times G_{i-1} \times \widetilde{G}_i \times G_{i+1} \times \cdots \times G_m$ obtained from G by applying the transformation of G_i to $\widetilde{G}_i$. Namely, G^{uv} is

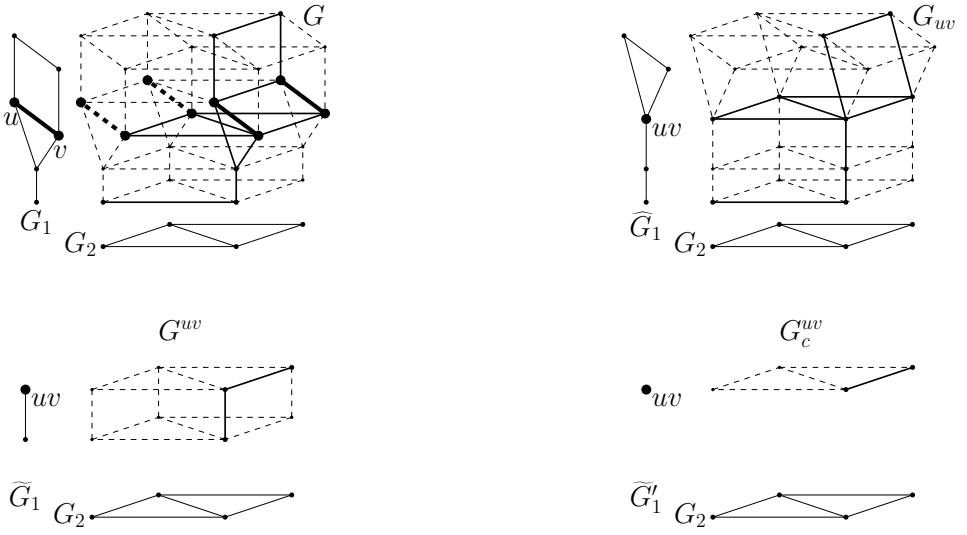


Figure 1: Examples of graphs G_{uv}, G^{uv} and G_c^{uv}.

the subgraph of $\widetilde{\Gamma}$ induced by the following set of vertices: (1) $(v_1, \ldots, v_{i-1}, \widetilde{w}, v_{i+1}, \ldots, v_m)$ is a vertex of G^{uv} if $(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_m)$ and $(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_m)$ are vertices of G and (2) $(v_1, \ldots, v_{i-1}, \widetilde{x}, v_{i+1}, \ldots, v_m)$ is a vertex of G^{uv} if $(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_m)$ is a vertex of G, $x \in N$, and $(v_1, \ldots, u, \ldots, v_m)$ and $(v_1, \ldots, v, \ldots, v_m)$ are vertices of G.

Notice that G_{uv} plays the same role in the Cartesian product of arbitrary graphs than in hypercubes (i.e., a vertex $(v_1, \ldots, w, \ldots, v_m)$ is in $V(G_{uv})$ if $(v_1, \ldots, u, \ldots, v_m)$ or $(v_1, \ldots, v, \ldots, v_m)$ belong to G). However, this is not exactly the case for G^{uv}. We define the graph G_c^{uv} which is the subgraph of G^{uv} induced by the vertices that have a central node $\widetilde{w}$ of $\widetilde{G}_i$ as their ith coordinate (see Fig. 1). If G_i is a K_2 (or, more generally, the edge uv does not belong to a triangle), then G^{uv} coincides with G_c^{uv}. The remaining vertices of G^{uv}, those having a leaf $\widetilde{x}$ of $\widetilde{G}_i$ as their ith coordinate, will be called *tip vertices*. We denote by $V_l(G^{uv})$ the set of *tip vertices*.

Let Γ be a Cartesian product of connected graphs $G_1, \ldots, G_m$ from $\mathcal{G}(H)$ and let G be an induced subgraph of Γ. Let G_i be any factor of Γ ($i \in \{1, \ldots, m\}$) and let uv be any edge of G_i. The four following lemmas are the key ingredients to an inductive proof of Theorem 1. For the proofs of these results and some applications to adjacency labeling schemes, see [1].

Lemma 1. $\text{VC-dim}(G) \leq \log n$.

Lemma 2. *Respectively computing the VC-dimension of G_{uv} and G_c^{uv} with respect to $\widehat{\Gamma}$ and $\widetilde{\Gamma}$ leads to the following inequalities*

$$\text{VC-dim}(G_{uv}) \leq \text{VC-dim}(G) \quad \textit{and} \quad \text{VC-dim}(G_c^{uv}) \leq \text{VC-dim}(G) - 1.$$

Lemma 3. *The graphs G, G_{uv}, G_c^{uv}, and G^{uv} satisfy the following relations:*

$$\begin{cases} |V(G)| = |V(G_{uv})| + |V(G^{uv})| - |V_l(G^{uv})| = |V(G_{uv})| + |V(G_c^{uv})|, \\ |E(G)| \leq |E(G_{uv})| + |E(G^{uv})| + |V(G_c^{uv})|. \end{cases}$$

Lemma 4. $\frac{|V(G^{uv})|}{|V(G_c^{uv})|} \leq |N|$, *where N is the set of the common neighbors of u and v.*

We have to prove that $\frac{|E(G)|}{|V(G)|} \leq \mu(H) \cdot \text{VC-dim}(G) \leq \mu(H) \cdot \log |V(G)|$. The second inequality follows from Lemma 1. We will prove the inequality $\frac{|E(G)|}{|V(G)|} \leq \mu(H)\,\text{VC-dim}(G)$ by induction on the number of vertices in the factors of Γ. Since each factor G_i of Γ belongs to $\mathcal{G}(H)$, G_i and any its subgraph contain a vertex v of degree at most $\mu(H)$. Let u be any neighbor of v in G_i. Then the set N of common neighbors of u and v has size at most $\mu(H)-1$. Consider the graphs G_{uv}, G^{uv}, and G^{uv}_c obtained from G by performing the previously described operations with respect to the edge uv of G_i. Then G_{uv} is a subgraph of the product $\widehat{\Gamma} = G_1 \times \cdots \times G_{i-1} \times \widehat{G}_i \times G_{i+1} \cdots \times G_m$. Since $\widehat{G}_i$ is a minor of G_i, all factors of $\widehat{\Gamma}$ belong to $\mathcal{G}(H)$. Moreover, since $\widehat{G}_i$ contains less vertices than G_i, we can apply the induction assumption to subgraphs of $\widehat{\Gamma}$, in particular to G_{uv}. Analogously, G^{uv} and G^{uv}_c are subgraphs of the product $\widetilde{\Gamma} = G_1 \times \cdots \times G_{i-1} \times \widetilde{G}_i \times G_{i+1} \times \cdots \times G_m$ and since $\widetilde{G}_i$ is a star isomorphic to a subgraph of G_i, all factors of $\widetilde{\Gamma}$ also belong to $\mathcal{G}(H)$. Since $\widetilde{G}_i$ contains less vertices than G_i, also do the graphs G^{uv} and G^{uv}_c. Consequently, we have $\frac{|E(G_{uv})|}{|V(G_{uv})|} \leq \mu(H) \cdot \text{VC-dim}(G_{uv})$ and $\frac{|E(G^{uv}_c)|}{|V(G^{uv}_c)|} \leq \mu(H)\,\text{VC-dim}(G^{uv}_c)$. Using the inequality $\frac{a_1+a_2}{b_1+b_2} \leq \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\}$ and Lemma 3, we obtain

$$\begin{aligned}\frac{|E(G)|}{|V(G)|} &\leq \frac{|E(G_{uv})| + |E(G^{uv})| + |V(G^{uv}_c)|}{|V(G_{uv})| + |V(G^{uv}_c)|} \\ &\leq \max\left\{\frac{|E(G_{uv})|}{|V(G_{uv})|}, \frac{|E(G^{uv})| + |V(G^{uv}_c)|}{|V(G^{uv}_c)|}\right\}.\end{aligned}$$

By Lemma 2, $\text{VC-dim}(G_{uv}) \leq \text{VC-dim}(G)$, whence

$$\frac{|E(G_{uv})|}{|V(G_{uv})|} \leq \mu(H)\,\text{VC-dim}(G_{uv}) \leq \mu(H)\,\text{VC-dim}(G).$$

Thus it remains to provide a similar upper bound for $\frac{|E(G^{uv})|+|V(G^{uv}_c)|}{|V(G^{uv}_c)|}$. Since $|E(G^{uv})|+|V(G^{uv}_c)| = |E(G^{uv}_c)|+|V_l(G^{uv})|+|V(G^{uv}_c)| = |E(G^{uv}_c)|+|V(G^{uv})|$ and $|N| \leq \mu(H)$, from Lemma 2 we conclude:

$$\begin{aligned}\frac{|E(G^{uv})| + |V(G^{uv}_c)|}{|V(G^{uv}_c)|} &= \frac{|E(G^{uv}_c)| + |V(G^{uv})|}{|V(G^{uv}_c)|} \\ &\leq \mu(H)\,\text{VC-dim}(G^{uv}_c) + \mu(H) \\ &\leq \mu(H)(\text{VC-dim}(G) - 1) + \mu(H) \\ &= \mu(H)\,\text{VC-dim}(G).\end{aligned}$$

This establishes the inequality $\frac{|E(G)|}{|V(G)|} \leq \mu(H)\,\text{VC-dim}(G)$ and concludes the proof of the theorem.

References

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