Distance Magic Labelings of Strongly Regular Graphs

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Abstract

A graph G on n vertices is said to be distance magic if there exists a bijection $f : V \to \{1, 2, \ldots, n\}$ and a constant k such that for any vertex x, $\sum_{y \in N(x)} f(y) = k$, where $N(x)$ is the set of all neighbours of x.

In this paper we utilize spectra of graphs to characterize strongly regular graphs admitting distance magic labelings and strongly regular graphs whose line graphs admitting distance magic labelings. In addition, we obtain necessary conditions for the existence of distance magic labeling of distance regular graphs of diameter 3.

1 Introduction

The notion of distance magic labeling was introduced separately in the PhD thesis of Vilfred [9] in 1994 and an article by Miller *et. al* [6] in 2003. A *distance magic labeling* of a graph G on n vertices is a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ with the property that there exists a *magic constant* k such that at any vertex x, its *weight* $w(x) = \sum_{y \in N(x)} f(y) = k$, where $N(x)$ is the open neighborhood of x, i.e., the set of vertices adjacent to x. A graph admitting a distance magic labeling is then called a *distance magic graph*. In 1999, Jinah [5] introduced a variation of distance magic labeling called the *closed distance magic labeling*, where the weight of a vertex x is summed over the closed neighborhood of x, i.e., the set containing x and all vertices adjacent to x.

It was proven in [9] that every graph is a subgraph of a distance magic graph, which showed that there is no forbidden subgraph characterization for distance magic graph. One of the most important results for distance magic labeling is in [7, 3] where it is shown that for a particular graph, the magic constant is unique and is determined by its fractional domination number. For more results, refer to survey articles in [2] and [8].

Several previous results are essential in proving our results and we shall list them here. By $H_{n,p}$ we mean the complete multipartite graph with p partite sets, each partite set consists of n vertices.

Theorem 1. *[6] For $n > 1$ and $p > 1$, $H_{n,p}$ has distance magic labeling if only if n is even or both n and p are odd.*

Theorem 2. *[1] If G is a regular graph and has close distance magic labeling, then -1 is an eigen value of G.*

Theorem 3. *[5] G has distance magic labeling if and only if $\overline{G}$ has closed distance magic labeling.*

2 Spectra of Graphs and Distance Magic Labelings

Let us consider a graph G admitting a distance magic labeling f with magic constant k. Then the following matrix equation holds.

$$A\mathbf{f} = k\mathbf{1}, \tag{1}$$

where A is the adjacency matrix of G, $\mathbf{f}$ is a vector of all labels under the labeling f, and $\mathbf{1}$ is a vector of all ones.

Equation (1) is essential in finding the following necessary condition for the existence of distance magic regular graphs.

Theorem 4. *If G is a regular graph admitting a distance magic labeling then 0 is an eigen value of G.*

Proof. Let G be an r-regular graph. Thus in addition to the matrix equation (1), another matrix equation holds, i.e., $A\mathbf{h} = k\mathbf{1}$, where $\mathbf{h} = (\frac{k}{r}\ \frac{k}{r}\ \ldots\ \frac{k}{r})^t$. Hence, $A(\mathbf{f} - \mathbf{h}) = 0$, which means 0 is an eigen value of G. □

Combining Theorem 4 with Theorems 2 and 3 we obtain the following.

Corollary 1. *If G is a regular graph and $\overline{G}$ is distance magic then -1 is an eigen value of G.*

A necessary condition for line graphs of regular graphs to admit distance magic labelings is presented in the following.

Theorem 5. *Let G be an r-regular graph. If the line graph of G, $L(G)$, is distance magic, then $G \approx K_2$ or $2 - r$ is an eigen value of G.*

Proof. Let the eigen values of G be $r, \lambda_1, \ldots, \lambda_{s-1}$, then the eigen values of $L(G)$ are $2r - 2, r - 2 + \lambda_1, \ldots, r - 2 + \lambda_{s-1}, -2$. Since the line graph of regular graph is also regular, then, if $L(G)$ has distance magic labeling, by Theorem 4, 0 is an eigen value of $L(G)$. Therefore, either $2r - 2 = 0$ or there exists an i such that $r - 2 + \lambda_i = 0$. If $2r - 2 = 0$ then $r = 1$ and $G \approx K_2$. In the second case, $2 - r$ is an eigen value of G. □

3 Distance Magic Labelings of Strongly Regular Graphs

Applying Theorem 4 to strongly regular graphs leads to the complete characterization of distance magic strongly regular graphs.

Theorem 6. *Let G be a strongly regular graph of degree r, on n vertices. G admits a distance magic labeling if and only if G is a regular complete multipartite graph $H_{n-r,\frac{n}{n-r}}$, where $n - r$ is even or both $n - r$ and $\frac{n}{n-r}$ are odd.*

Proof. By Theorem 4, G has 0 as an eigenvalue. It is well known that a strongly regular graph has eigenvalue 0 if and only if it is a complete multipartite graph (see, for instance, [4]). Since G is r-regular, then the cardinality of each partite set is $n - r$, and so $G \approx H_{n-r,\frac{n}{n-r}}$. We complete the proof by applying Theorem 1. □

We also obtain the following characterisation of strongly regular graphs whose line graphs are distance magic.

Theorem 7. *Let G be a strongly regular graph. The line graph of G is distance magic if and only if G is a cycle of order 4.*

Proof. Let G be a strongly regular graph of order n with parameter (r, a, c), where $c \geq 1$ and $0 \leq a < r$. Let the eigen values of G be r, x_1, x_2, where x_1 and x_2 are the roots of $x^2+(c-a)x+(c-r) = 0$. Consequently, the eigen values of $L(G)$ are $2r-2, r-2+x_1, r-2+x_2, -2$. If $L(G)$ is distance magic, then by Theorem 4, either $2r-2 = 0, r-2+x_1 = 0$, or $r-2+x_2 = 0$.

We shall consider two cases: $r = 2$ and $r > 2$.

If $r = 2$, without loss of generality, $r-2+x_1 = 0$, and so $x_1 = 0$, which means G is a complete multipartite graph. Since $x_1x_2 = c-r$, $c = r = 2$, and so $G \approx C_4$, where its line graph, $L(C_4) \approx C_4$ is distance magic.

If $r > 2$, then without loss of generality, $r-2+x_1 = 0$, and $x_1 = 2-r$. Since $x_1+x_2 = -(c-a)$, then $x_2 = a-c-2+r$. From $x_1x_2 = c-r$, we obtain

$$(r-2)(r-c+a-2) = r-c \tag{2}$$

If $r-c+a-2 = 0$, then $c = r$ follows from (2), and so $a = 2$. Therefore, G has parameter $(r, 2, r)$ or $G \approx K_{2,2,2}$. If $r-c+a-2 \neq 0$, from equation (2) we obtain $r-c \neq 0$. Since $r \geq c$ then $r-c > 0$, and so $r-c+a-2 > 0$. Applying again equation (2), $r-c \geq r-2$, which means $c \leq 2$. The case of $c = 1$ leads to G having parameter $(3, 2, 1)$. On the other hand, the case of $c = 2$, leads to G having parameter $(3, 2, 2), (4, 1, 2)$, or $(5, 0, 2)$. Parameters $(3, 2, 1)$ and $(3, 2, 2)$ are not possible, where parameters $(4, 1, 2)$ and $(5, 0, 2)$ give two graphs: the Hamming graph $H(2, 3)$ and the folded 5-cube graph.

Our computational approach of utilizing an exhaustive search of distance magic labelings showed that the line graphs of the three remaining graphs: the complete multipartite graph $K_{2,2,2}$, the Hamming graph $H(2, 3)$, and the folded 5-cube graph do not admit distance magic labelings. □

Similar to the case of distance magic labelings, we could also show that line graphs of strongly regular graphs do not admit closed distance magic labelings.

Theorem 8. *The line graphs of strongly regular graphs do not admit closed distance magic labeling.*

Proof. Let G be a strongly regular graph with parameter (r, a, c). Suppose that $L(G)$ admits a closed distance magic labeling. Applying Theorem 2 to the spectra of $L(G)$ leads to

$$(r-1)(c+2-r-a) = c-1. \tag{3}$$

If $c = 1$, then $(r-1)(3-r-a) = 0$. Since $r \geq 2$, then $r-1 \neq 0$ and $r+a = 3$. Thus G has parameters $(2, 1, 1)$ or $(3, 0, 1)$. The first parameter is not possible and the second parameter gives the Petersen graph. However our computational approach showed that the line graph of Petersen graph is not closed distance magic.

If $c \geq 2$, then $c+2-r-a \geq 1$. From Equation (3), $r-1 \leq c-1$ which gives $r \leq c$. Since $r \geq c$ then $r = c$ and $a = 1$, which means G has parameter $(r, 1, r)$ and is also a regular complete multipartite graph, which is impossible to happen. □

We shall conclude by presenting a necessary condition for the existence of distance magic labelings for distance regular graphs of larger diameter, in particular for diameter 3. Whether the necessary condition is also sufficient will be left as a future work.

Theorem 9. *Let G be a distance regular graph with diameter d and intersection array*

$$\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}.$$

If G is distance magic then

$$det \begin{pmatrix} a_2 & c_3 & & & 0 \\ b_2 & a_3 & c_4 & & \\ & b_3 & a_4 & \ddots & \\ & & & \ddots & c_d \\ 0 & & & & a_d \end{pmatrix} = 0,$$

where $a_i = k - b_i - c_i, i \in \{1, 2, \ldots, d\}$, and $c_1 = 1$.

Proof. The eigen values of G are also the eigen values of the matrix

$$B = \begin{pmatrix} 0 & 1 & & & & 0 \\ k & a_1 & c_2 & & & \\ & b_1 & a_2 & c_3 & & \\ & & b_2 & a_3 & \ddots & \\ & & & & \ddots & c_d \\ 0 & & & & & a_d \end{pmatrix},$$

where $a_i = k - b_i - c_i, i \in \{1, 2, 3, \ldots, d\}$ and $c_1 = 1$. Since 0 is an eigen value of G then $det(B - 0 \cdot I) = 0$, where I is an identity matrix. This completes the proof. □

By combining the previous theorem with the following known equations for distance regular graph of diameter 3 and intersection array $\{k, b_1, b_2; 1, c_2, c_3\}$: $k = b_1 + 1 + a_1$, $k = b_2 + c_2 + a_2$, $k = c_3 + a_3$, $c_1 = 1, b_0 = k, a_0 = 0$, we obtain the following.

Corollary 2. *Let G be a distance regular graph of diameter 3, with intersection array $\{k, b_1, b_2; 1, c_2, c_3\}$. If G is distance magic then $(k - b_2 - c_2)(k - c_3) = b_2c_3$.*

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