About the Number of Oriented Hamiltonian Paths and Cycles in Tournaments

Zeina Ghazo Hanna — **Lebanese University**
Amine El Sahili — **Lebanese University**

Abstract

We prove that a tournament T and its complement $\overline{T}$ contain the same number of oriented Hamiltonian paths (resp. cycles) of any given type, as a generalization of Rosenfeld's result proved for antidirected paths.

An oriented Hamiltonian path in a tournament is an oriented path containing all its vertices, and if this path is directed, then it is said to be a directed Hamiltonian path. The definitions are similar for cycles. The first result of existence concerns the directed paths. It was given by Redei [1] in 1934, who proved that every tournament contains a directed Hamiltonian path. Later on, in 1969, Grunbaüm [2] treated the existence of antidirected paths, which are paths where any two consecutive arcs have opposite directions: he proved that at the exception of 3 cases, which are the directed 3-cycle, the regular tournament on 5 vertices, and the Paley tournament on 7 vertices, every tournament contains an antidirected Hamiltonian path. Later on, many researches followed on this subject until the problem of existence was completely setteled by Havet and Thomassé [3] in 1998: they proved that at the exception of the 3 cases previously mentioned by Grunbaüm, every tournament contains any oriented Hamiltonian path. Concerning the existence of oriented cycles, Havet [4] proved in 1999 that every tournament contains any oriented Hamiltonian cycle, except possibly the Hamiltonian circuit (directed cycle) in a reducible tournament, which is a tournament that is not strong.

Let's consider the definitions introduced by El Sahili and Abi Aad in their article [6] "Antidirected Hamiltonian paths and directed cycles in tournaments".
Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s);\ s \geq 1,\ \alpha_i \in \mathbb{Z},\ \alpha_i \cdot \alpha_{i+1} < 0,\ \forall\ i = 1, \ldots, s-1$. An oriented path P is said to be of type $P(\alpha_1, \alpha_2, \ldots, \alpha_s)$ if P is formed by s consecutive blocks (i.e. maximal directed subpaths) $I_1, I_2, \ldots, I_s$ such that $length(I_i) = \mid I_i \mid = \mid \alpha_i \mid$ with $\alpha_i > 0$ if the arcs of the block of length $\mid \alpha_i \mid$ are forward arcs, and $\alpha_i < 0$ if the arcs of the block are backward. We have a similar definition for cycles: Let $\alpha = (\alpha_1, \ldots, \alpha_s);\ \alpha_i \in \mathbb{Z}$, s is even if $s > 1$, $\alpha_i \cdot \alpha_{i+1} < 0,\ \forall\ i = 1, \ldots, s-1,\ \alpha_s \cdot \alpha_1 < 0$. An oriented cycle C is said to be of type $C(\alpha_1, \ldots, \alpha_s)$ if C is formed by s consecutive blocks $I_1, I_2, \ldots I_s$, with $\mid I_i \mid = \mid \alpha_i \mid$ and $\alpha_i > 0$ if the arcs of the block are forward, and $\alpha_i < 0$ otherwise. If T is a tournament, then $\mathcal{P}_T(\alpha_1, \ldots, \alpha_s)$ (resp. $\mathcal{C}_T(\alpha_1, \ldots, \alpha_s)$) is defined as being the set of oriented paths (resp. cycles) in T of type $P(\alpha_1, \ldots, \alpha_s)$ (resp. $C(\alpha_1, \ldots, \alpha_s)$) and $f_T(\alpha_1, \ldots, \alpha_s)$ (resp. $g_T(\alpha_1, \ldots, \alpha_s)$) represents the cardinal of this set.
For $\alpha = (\alpha_1, \ldots, \alpha_s)$ in $\mathbb{Z}^s$, we denote by $-\alpha$ the uplet $(-\alpha_1, \ldots, -\alpha_s)$ and by $\overline{\alpha}$ the uplet $(\alpha_s, \alpha_{s-1} \ldots, \alpha_1)$. A uplet α is said to be symmetric if $\alpha = -\overline{\alpha}$. An oriented path P (resp. cycle C) is said to be symmetric if there exists a symmetric uplet α such that P (resp. C) is of type $P(\alpha)$ (resp. $C(\alpha)$). Otherwise, the path P (resp. cycle C) is non-symmetric.
Also, a notion of period has been introduced: Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$. An integer $1 \leq r \leq s$ is said to be a period of α if $[i \equiv j\ (mod\ r) \Rightarrow \alpha_{i_s} = \alpha_{j_s}]$ where i_s is the unique integer belonging to $\{1, 2, \ldots, s\}$ such that $i \equiv i_s\ (mod\ s)$. Let $r(\alpha) = min\{r;\ r \text{ is a period of } \alpha\}$. We can prove that

r is a period of $\alpha \iff r(\alpha)$ divides r, and as a result $r(\alpha)$ divides s, since s is a trivial period of α, and we set $t(\alpha) = \frac{s}{r(\alpha)}$.

Rosenfeld [5] proved in 1974 that the number of antidirected Hamiltonian paths starting by a forward arc is equal to the number of antidirected Hamiltonian paths starting by a backward arc, in any tournament, i.e. $f_T(1,-1,1,...) = f_T(-1,1,-1,...)$, which can be stated as the following: The number of anidirected Hamiltonian paths in a tournament T is equal to the number of those in its complement $\overline{T}$.
During our work, we could first generalize Rosenfeld's result for any type of oriented Hailtonian paths, by proving the following theorem:

Theorem 1. *Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i.\alpha_{i+1} < 0$, $\alpha_1 \geq 0$, and let T be a tournament of order n; $n = \sum_{i=1}^{s} \mid \alpha_i \mid +1$. We have:*

$$f_T(\alpha) = f_T(-\alpha).$$

The proof of this theorem is based on set's theory, and requires a double induction on the number s of components of the uplet α, and on the cardinal $|\alpha_1|$.
The above result can be stated as the following: "Every tournament and its complement contain the same number of oriented Hamiltonian paths of any given type", which is in fact a generalization of Rosenfeld's result.

We could also establish this fact for cycles. To prove it, we first had to find a link between the number of oriented Hamiltonian paths of a given type in a tournament, and the number of some particular corresponding oriented cycles (defined in what follows):
Let T be a tournament on n vertices. An oriented cycle C in T is said to be generated by an oriented path $P = x_1x_2 \ldots x_r$ if $C = P \cup < \{x_1, x_r\} >$. We write $C = C_P$. We define the relation $\mathcal{R}$ on the set of oriented paths in T by $P\mathcal{R}P' \iff C_P = C_{P'}$, which is obviously an equivalence relation, and so is $\mathcal{R}_\alpha$, the restriction of $\mathcal{R}$ on the set $\mathcal{P}_T(\alpha)$.
Now given a Hamiltonian cycle of a certain type $C(\beta)$, it can be generated by many Hamiltonian paths of the same type, say $P(\alpha)$, and we can compute their exact number using the notion of period. In fact we proved the following proposition:

Proposition 1. *Let $P \in \mathcal{P}_T(\alpha)$ be an oriented Hamiltonian path in a tournament T of order n and let C_P be the cycle generated by P in T, of type $C(\beta)$, and let $\overline{P}$ be the equivalence class of P with respect to $\mathcal{R}_\alpha$. Then if C_P is non-symmetric, such that C_P has at least 2 blocks (i.e. C_P is not a circuit), then $\mid \overline{P} \mid = t(\beta)$, while if C_P is symmetric, then $\mid \overline{P} \mid = 2.t(\beta)$. Moreover, if C_P is a circuit, then $\mid \overline{P} \mid = n$.*

On the other hand, remark that given an oriented path $P = x_1x_2...x_r$ of a certain type $P(\alpha)$ in a tournament T, then the type of the cycle C_P in T can take two different values, $C(\beta)$ or $C(\beta')$, and it depends on whether (v_1, v_r) or (v_r, v_1) is an arc in T.
We proved the following proposition:

Proposition 2. *Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$, $\alpha_i.\alpha_{i+1} < 0$, and let T be a tournament of order $n = \sum_{i=1}^{s} \mid \alpha_i \mid +1$. We have:*

$$\mathcal{C}_T(\beta) = \mathcal{C}_T(\beta') \iff \alpha \text{ is symmetric},$$

where $C(\beta)$ and $C(\beta')$ are the two types of Hamiltonian cycles in T that can be generated by a Hamiltonian path of type $P(\alpha)$ in T.

Given a uplet $\beta = (\beta_1, \beta_2, \ldots, \beta_s) \in \mathbb{Z}^s$; s is even, and $\beta_i\beta_{i+1} < 0,\ \forall\ i = 1, \ldots, s-1$, then $\forall$ $1 \leq i \leq s$, define $\beta_i * 1$ as

$$\beta_i * 1 = \begin{cases} \beta_i - 1 & \text{if} \quad \beta_1 > 0 \\ \beta_i + 1 & \text{if} \quad \beta_1 < 0 \end{cases}.$$

Using the two previous propositions, we could establish the following theorem:

Theorem 2. *Let T be a tournament of order n, and $(\beta_1, \ldots, \beta_s) \in \mathbb{Z}^s; \sum_{i=1}^{s} \mid \beta_i \mid = n$, s is even and $\beta_i\beta_{i+1} < 0,\ \forall\ i = 1, \ldots, s-1$. Then:*

If $(\beta_1 * 1, \beta_2, \ldots, \beta_s)$ is symmetric, we have:

$$f_T(\beta_1 * 1, \beta_2, \ldots, \beta_s) = g_T(\beta_1, \beta_2, \ldots, \beta_s).t(\beta_1, \beta_2, \ldots, \beta_s).$$

Otherwise we have:

$$\begin{aligned} f_T(\beta_1 * 1, \beta_2, \ldots, \beta_s) = \quad & \delta(\beta_1, \beta_2, \ldots, \beta_s).g_T(\beta_1, \beta_2, \ldots, \beta_s).t(\beta_1, \beta_2, \ldots, \beta_s) \\ & + \delta(\beta_1 * 1, \beta_2, \ldots, \beta_s * 1).g_T(\beta_1 * 1, \beta_2, \ldots, \beta_s * 1).t(\beta_1 * 1, \beta_2, \ldots, \beta_s * 1), \end{aligned}$$

$$\text{where } \delta(\gamma) = \begin{cases} 1 & \text{if} \quad \gamma \text{ non} - \text{symmetric and is not a circuit} \\ 2 & \text{if} \quad \gamma \text{ is symmetric} \\ \frac{n}{t(\gamma)} & \text{if} \quad \gamma \text{ is a circuit} \end{cases}.$$

(Note that $C(\beta_1, \beta_2, \ldots, \beta_s)$ and $C(\beta_1 * 1, \beta_2, \ldots, \beta_s * 1)$ are the two types of cycles that can be generated by a path of type $P(\beta_1 * 1, \beta_2, \ldots, \beta_s)$).

Finally, Theorems 1 and 2 allowed us to prove our result for cycles:

Theorem 3. *Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i.\alpha_{i+1} < 0$, $\alpha_1 \geq 0$, s is even if $s \neq 1$, and let T be a tournament or order n; $n = \sum_{i=1}^{s} \mid \alpha_i \mid$. We have:*

$$g_T(\alpha) = g_T(-\alpha).$$

At last, we could go further by noticing that every digraph of maximal degree $\Delta \leq 2$ is a union of oriented cycles and paths, and we proved the following:

Theorem 4. *Let T be a tournament and let D be a digraph of maximal degree $\Delta(G(D)) \leq 2$. Then the number of copies of D in T and in its complement $\overline{T}$ is the same.*

However, as this result is not always true for digraphs of maximal degree $\Delta > 2$, we ask the following question: Let $f_T(D)$ be the number of copies of a digraph D in a tournament T,

Problem 1. *Can we characterize the set $\mathcal{H}$ of all digraphs D such that $f_T(D) = f_{\overline{T}}(D)$ for a tournament T?*

References

[1] L. Redei, Ein kombinatorischer satz, Acta Litt. Szeged, (1934), no.7, 39-43.

[2] B. Grunbaum, Antidirected hamiltonian paths in tournaments, Journal of Combinatorial Theory (1969).

[3] F. Havet and S. Thomasse, Oriented hamiltonian paths in tournaments: a proof of Rosenfeld's conjecture, Journal of Combinatorial Theory (1998), no. Series B 78, 243-273.

[4] F. Havet, Oriented Hamiltonian cycles in tournaments, Journal of Combinatorial Theory (1999), no. 1-31.

[5] M. Rosenfeld, Antidirected Hamiltonian Circuits in Tournaments, Journal of Combinatorial Theory (B) 16 (1974), 234-242.

[6] A. El Sahili and M. Abi Aad, Antidirected Hamiltonian paths and directed cycles in tournaments, Accepted.