On a Conjecture regarding Identification in the Hamming Graphs

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Abstract

Identifying codes were introduced by Karpovsky *et al.* (1998) and have been widely studied since then. Goddard and Wash (2013) studied them in the Cartesian products of complete graphs and provided a conjecture for three equal complete graphs stating that $\gamma^{ID}(K_q \Box K_q \Box K_q) = q^2$. In this article, we present an infinite family of counterexamples to this conjecture showing that $\gamma^{ID}(K_q \Box K_q \Box K_q) \leq q^2 - q/4$ when q is a power of four. Goddard and Wash also showed a lower bound $\gamma^{ID}(K_q \Box K_q \Box K_q) \geq q^2 - q\sqrt{q}$. We improve this lower bound to $\gamma^{ID}(K_q \Box K_q \Box K_q) \geq q^2 - \frac{3}{2}q$. Moreover, we also show that $\gamma^{ID}(K_q^n) \leq q^{n-k}$ for $n = 3\frac{q^k-1}{q-1}$ improving the previous upper bound $\gamma^{ID}(K_q^n) \leq q^{n-1}$ by Goddard and Wash.

Let us assume that the graph $G = (V, E)$ is undirected, simple and finite. The set of vertices adjacent to a vertex v is called the *open neighbourhood* of v, denoted by $N(v)$, and the set $N(v) \cup \{v\} = N[v]$ is called the *closed neighbourhood* of v. A nonempty subset C of the vertex set V is called a *code* and its elements are called *codewords*. We define *the identifying set* or *the I-set* of a vertex u as

$$I(G, C; u) = I(C; u) = I(u) = N[u] \cap C$$

depending on whether the code or the graph is clear from the context. The set of vertices C is called a *dominating set* if $I(v) \neq \emptyset$ for each vertex $v \in V$ and the minimum size of a dominating set in a graph G is called the *domination number* $\gamma(G)$. Karpovsky, Chakrabarty and Levitin defined *identifying codes* in [5]— for other articles on the subject, see the site [6]. For the complete proofs of the theorems and lemmas of this paper and some additional results, see [4].

Definition 1. A code $C \subseteq V$ is *identifying* in a graph $G = (V, E)$ if C is a dominating set and

$$I(u) \neq I(v)$$

for each pair of distinct vertices $u, v \in V$. An identifying code C of minimum cardinality in a finite graph G is called *optimal* and its cardinality is denote by $\gamma^{ID}(G)$.

A graph is called the *complete graph* on q vertices, denoted by K_q, if each pair of vertices of the graph is adjacent. The *Cartesian product* of two graphs $K_1 = (V_1, E_1)$ and $K_2 = (V_2, E_2)$ is defined as $K_1 \Box K_2 = (V_1 \times V_2, E)$, where E is a set of vertices such that $(u_1, u_2)(v_1, v_2) \in E$ if and only if $u_1 = v_1$ and $u_2 v_2 \in E_2$, or $u_2 = v_2$ and $u_1 v_1 \in E_1$. The Cartesian product of n copies of K_q is denoted by K_q^n. Moreover, we denote $V(K_q^3) = \{(x, y, z) \mid (x, y, z) \in \mathbb{Z}^3, 1 \leq x, y, z \leq q\}$ and $(x, y, z)(x', y', z') \in E(K_q^3)$ if exactly one of the three coordinates differ. Furthermore, we say that $P \subseteq V(K_q^3)$ is a *pipe* if it is fixes two out of three coordinates. For example, $\{(1, 1, z) \mid 1 \leq z \leq q\}$ is a pipe.

Previously, identifying codes have been studied in $K_q \Box K_m$ in [2] and [3]. Goddard and Wash (in [2]) studied identification also in $K_q \Box K_m \Box K_l$ and gave a conjecture for the cardinality of an optimal identifying code in K_q^3.

Conjecture 2 ([2])**.** For all $q \geq 1$, $\gamma^{ID}(K_q \Box K_q \Box K_q) = q^2$.

	3, 4	1	2
2, 4	4	2	
1	2	3	3
	4		1, 3

Figure 1: Identifying code C^1 in K_4^3.

	3	4	2
4		1	3
2	4		1
3	1	2	

Figure 2: Identifying code C_L in $K_4^3[V(K_4^3) \setminus Di]$.

In [2], Goddard and Wash prove that $\gamma^{ID}(K_q^3) \leq q^2$ for all $q \geq 1$. Moreover, by an exhaustive computer search, they show that $\gamma^{ID}(K_q^3) = q^2$ when $q = 3$. Furthermore, they provide a lower bound stating that $\gamma^{ID}(K_q^3) \geq q^2 - q\sqrt{q}$. Recall that for the domination number we have $\gamma(K_q^3) = \left\lceil \frac{q^2}{2} \right\rceil$ (see [1]).

Lemma 3. *Let C be a code in K_q^3 and v be a vertex of K_q^3.*

(i) If a vertex v has two codewords in its I-set and they do not locate within a single pipe, then there is exactly one other vertex which has those two codewords in its I-set.

(ii) The I-set $I(v)$ is not a subset of any other I-set if and only if there are at least three codewords in $I(v)$ and they do not locate within a single pipe.

Goddard and Wash [2] gave the following construction for identifying codes of cardinality q^2.

Lemma 4. *The code $C_q = \{(a, b, c) \mid a+b+c \equiv 0 \pmod q\}$ is identifying in K_q^3 with the additional property that each pipe in K_q^3 has exactly one codeword.*

Proof. There is exactly one codeword in each pipe since if we fix two of the three coordinates, then for exactly one value of the third coordinate the equation $a + b + c \equiv 0 \pmod q$ is satisfied. □

The identifying code in K_4^3 of the following theorem is of cardinality 15. The code is presented in Figure 1; for example, $(2, 1, 3)$ corresponds to the number 3 in the top row and second column.

Theorem 5. *The code $C^1 = \{(2,1,3), (2,1,4), (3,1,1), (4,1,2), (1,2,2), (1,2,4), (2,2,4), (3,2,2), (1,3,1), (2,3,2), (3,3,3), (4,3,3), (2,4,4), (4,4,1), (4,4,3)\}$ is identifying in $K_4 \Box K_4 \Box K_4$.*

In what follows, we call the set $Di = \{(j, j, j) \mid 1 \leq j \leq 4\}$ as a *diagonal*. We also need another code, C_L, to produce the infinite family of codes. The code is represented in Figure 2 for the graph $K_4^3[V(K_4^3) \setminus Di]$, that is, for the graph K_4^3 with its diagonal vertices Di deleted.

Lemma 6. *The code $C_L = \{(2,1,3), (3,1,4), (4,1,2), (1,2,4), (3,2,1), (4,2,3), (1,3,2), (2,3,4), (4,3,1), (1,4,3), (2,4,1), (3,4,2)\}$ is identifying in $K_4^3[V(K_4^3) \setminus Di]$ and for each codeword $c \in C_L$ we have $I(c) = \{c\}$.*

Due to the recursive nature of our construction we define an operation which combines two codes in K_q^3 and K_m^3 into a code in K_{qm}^3.

Definition 7. Let $C_1 \subseteq \{(x, y, z) \mid 1 \leq x, y, z \leq q\}$ and $C_2 \subseteq \{(x, y, z) \mid 1 \leq x, y, z \leq m\}$ be codes in K_q^3 and K_m^3, respectively. Define $\mathrm{Ext}(C_1, C_2) = \{(x, y, z, a, b, c) \mid (x, y, z) \in C_1, (a, b, c) \in C_2\}$.

The sextuple of coordinates can be interpreted in the following way.

Observation 8. We can interpret each vertex $(v, u, w) \in K^3_{qm} = \{(x', y', z') \mid 1 \leq x', y', z' \leq qm\}$ as (x, y, z, a, b, c) where $1 \leq x, y, z \leq q$ and $1 \leq a, b, c \leq m$ by having $v = x + q(a-1)$, $u = y + q(b-1)$ and $w = z + q(c-1)$. Hence, we can consider the graph K^3_{qm} consisting of m^3 subcubes each consisting of q^3 vertices. Moreover, the three latter coordinates, (a, b, c), inform the location of the subcube and the three former coordinates tell the location of the vertex within the subcube.

Theorem 9. *The code $C^t = Ext(C_{q/4}, C_L) \cup Ext(C^{t-1}, Di)$ is identifying in $K_q \Box K_q \Box K_q$, where $q = 4^t$ and $t \geq 2$. This gives $\gamma^{ID}(K_q \Box K_q \Box K_q) \leq q^2 - \frac{q}{4}$.*

Proof. By induction, we get $|C^t| = |C_L| \cdot |C_{q/4}| + |Di| \cdot |C^{t-1}| = \frac{3}{4}q^2 + 4|C^{t-1}| = q^2 - \frac{q}{4}$. The basic idea behind the code is dividing K^3_q into 64 *subcubes* and then identifying the location of the subcube with the codes Di and C_L and the location within the subcube with the codes $C_{q/4}$ and C^{t-1}. This is possible due to the property shown in Lemma 4. Moreover, $\mathrm{Ext}(C^{t-1}, Di)$ identifies vertices locating in the diagonal subcubes and $\mathrm{Ext}(C_{q/4}, C_L)$ identifies vertices locating in the other subcubes. □

So far, we have given constructions for identifying codes in K^3_q with $q = 4^t$. However, we can further use these codes to construct new identifying codes for other values of q.

Theorem 10. *If $2 \cdot 4^t \leq q \leq 2 \cdot 4^{t+1} - 1$, then we have*

$$\gamma^{ID}(K_q \Box K_q \Box K_q) \leq q^2 - 4^{t-1}.$$

Goddard and Wash have shown that $\gamma^{ID}(K^3_q) \geq q^2 - q\sqrt{q}$ for $q \in \mathbb{Z}$. In what follows, we show a new lower bound: $\gamma^{ID}(K^3_q) \geq q^2 - \frac{3}{2}q$ for $q \in \mathbb{Z}$.

Definition 11. Let C be a code in K^3_q and i and j be integers such that $1 \leq i \leq q$ and $1 \leq j \leq 3$. Define a *layer* of the graph K^3_q, denoted by D^j_i, as the set of vertices which fixes the jth coordinate as i, i.e., $D^2_i = \{(x, i, z) \mid 1 \leq x, z \leq q\}$. Define also $C^j_i = C \cap D^j_i$, $X^j_i = \{v \in D^j_i \mid I(C^j_i; v) = \emptyset\}$ and $X = \bigcup_{j=1}^{3} \bigcup_{i=1}^{q} X^j_i$.

We say that a codeword c is a *corner* if it has at least two other codewords in its I-set not belonging to a same subcube. Moreover, if those codewords are in the layer D^j_i, then we say that c is a corner of the layer D^j_i and denote with k^j_i the number of corners of the layer D^j_i. A codeword which is not a corner or an isolated codeword is a *fellow*. Let $M^j_i \subseteq D^j_i$ be a minimum dominating set of D^j_i such that $C^j_i \subseteq M^j_i$. Then we denote $f^j_i = |M^j_i| - q$ and $a^j_i = q - |C^j_i|$. Note that $\gamma(K_q \Box K_q) = q$.

Lemma 12. *If C is an identifying code in K^3_q, then we have*

$$\sum_{j=1}^{3} \sum_{i=1}^{q} (a^j_i + f^j_i)^2 \leq 6 \sum_{j=1}^{3} \sum_{i=1}^{q} f^j_i.$$

Proof. Note that we need $q + f^j_i$ codewords to dominate the layer D^j_i. Hence, the value f^j_i can be understood as a measurement of how much the structure of the code within a layer increases the cardinality of X^j_i. Indeed, observe that if a non-codeword in the layer D^j_i is not dominated by C^j_i, then it belongs to X^j_i and there is a row and a column in the layer D^j_i without codewords.

Moreover, observe that we have only $q - a_i^j$ codewords in the layer D_i^j. Hence, there are at least $q + f_i^j - (q - a_i^j) = f_i^j + a_i^j$ rows and columns which do not have a codeword when $f_i^j + a_i^j \geq 0$. Thus, we have $(a_i^j + f_i^j)^2 \leq |X_i^j|$ and hence, $\sum_{j=1}^{3} \sum_{i=1}^{q} (a_i^j + f_i^j)^2 \leq |X|$.

For each vertex $v \in X$ we have $c \in I(v)$ for some $c \in C$. Moreover, we have $c' \in I(c)$ since otherwise $I(v) = I(c)$. Hence, c is a corner or a fellow. Moreover, we can show that if c is a fellow then there is a corner in its I-set. Thus, each vertex of X is associated to a corner. Furthermore, it can be shown that each corner can be associated to at most three vertices of X. Each corner is a corner of at least one layer and hence, the number of corners is at most $\sum_{j=1}^{3} \sum_{i=1}^{q} k_i^j$. Thus, we get $|X| \leq \sum_{j=1}^{3} \sum_{i=1}^{q} k_i^j$. Furthermore, we can show that $k_i^j \leq 2f_i^j$ and hence, we get $|X| \leq 6 \sum_{j=1}^{3} \sum_{i=1}^{q} f_i^j$ which gives the claim. □

Theorem 13. *We have* $\gamma^{ID}(K_q \Box K_q \Box K_q) \geq q^2 - \frac{3}{2}q$.

Proof. We obtain

$$\begin{aligned} |C| =& \frac{1}{3} \sum_{j=1}^{3} \sum_{i=1}^{q} (q - a_i^j) = q^2 - \frac{1}{3} \left(\sum_{j=1}^{3} \sum_{i=1}^{q} (a_i^j + f_i^j) - \sum_{j=1}^{3} \sum_{i=1}^{q} f_i^j \right) \\ \geq& q^2 - \frac{1}{3} \left(\sum_{j=1}^{3} \sum_{i=1}^{q} (a_i^j + f_i^j) - \frac{1}{6} \sum_{j=1}^{3} \sum_{i=1}^{q} (f_i^j + a_i^j)^2 \right) \geq q^2 - \frac{3}{2}q. \end{aligned}$$

The first inequality is due to Lemma 12 and the second one is gained using Lagrange's method and then finding the minimum of the resulting polynomial. □

Previously, Goddard and Wash also showed that $\gamma^{ID}(K_q^n) \leq q^{n-1}$ for $n, q \in \mathbb{Z}$ ([2]). We improve this significantly in the next theorem. Recall that, by [5], we get $\gamma^{ID}(K_q^{3n}) \geq \frac{2}{3}q^{3n-k}$ for $n = \frac{q^k-1}{q-1}$ (compare this lower bound to the second upper bound given in the following theorem).

Theorem 14. *Let* $n = \frac{q^k-1}{q-1}$ *for some prime power* q *and* $k \geq 1$. *Then*

$$\gamma^{ID}(K_q^n) \leq 3q^{n-k} \text{ and } \gamma^{ID}(K_q^{3n}) \leq q^{3n-k}.$$

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