On the maximum size of connected hypergraphs without a path of given length

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Abstract

In this note we asymptotically determine the maximum number of hyperedges possible in an r-uniform, connected n-vertex hypergraph without a Berge path of length k, as n and k tend to infinity. We show that, unlike in the graph case, the multiplicative constant is smaller with the assumption of connectivity.

1 Introduction

Let P_k denote a path consisting of k edges in a graph G. There are several notions of paths in hypergraphs the most basic of which is due to Berge. A Berge path of length k is a set of $k+1$ distinct vertices $v_1, v_2, \ldots, v_{k+1}$ and k distinct hyperedges $h_1, h_2, \ldots, h_k$ such that for $1 \leq i \leq k$, $v_i, v_{i+1} \in h_i$. A Berge path is also denoted simply as P_k, and the vertices v_i are called basic vertices. If $v_1 = v$ and $v_{k+1} = w$, then we call the Berge path a Berge v-w-path. A hypergraph $\mathcal{H}$ is called connected if for any $v \in V(\mathcal{H})$ and $w \in V(\mathcal{H})$ there is a Berge v-w-path. Let $N_s(G)$ denote the number of s-vertex cliques in the graph G.

A classical result of Erdős and Gallai [6] asserts that

Theorem 1 (Erdős-Gallai)**.** *Let G be a graph on n vertices not containing P_k as a subgraph, then*

$$|E(G)| \leq \frac{(k-1)n}{2}.$$

In fact, Erdős and Gallai deduced this result as a corollary of the following stronger result about cycles,

Theorem 2 (Erdős-Gallai)**.** *Let G be a graph on n vertices with no cycle of length at least k, then*

$$|E(G)| \leq \frac{(k-1)(n-1)}{2}.$$

Kopylov [4] and later Balister, Győri, Lehel and Schelp [3] determined the maximum number of edges possible in a connected P_k-free graph.

Theorem 3. *Let G be a connected n-vertex graph with no P_k, $n > k \geq 3$. Then $|E(G)|$ is bounded above by*

$$\max\{\binom{k-1}{2} + n - k + 1, \binom{\lceil \frac{k+1}{2} \rceil}{2} + \left\lfloor \frac{k-1}{2} \right\rfloor (n - \left\lceil \frac{k+1}{2} \right\rceil)\}.$$

Observe that, although the upper bound is lower in the connected case, it is nonetheless the same asymptotically. Balister, Győri, Lehel and Schelp also determined the extremal cases.

Definition 1. *The graph $H_{n,k,a}$ consists of 3 disjoint vertex sets A, B, C with $|A| = a$, $|B| = n-k+a$ and $|C| = k-2a$. $H_{n,k,a}$ contains all edges in $A \cup C$ and all edges between A and B. B is taken to be an independent set. The number of s-cliques in this graph is*

$$f_s(n,k,a) = \binom{k-a}{s} + (n-k+a)\binom{a}{s-1}.$$

The upper bound of Theorem 3 is attained for the graph $H_{n,k,1}$ or $H_{n,k,\lfloor \frac{k-1}{2} \rfloor}$.

We now mention some recent results of Luo [5] which will be essential in our proof.

Theorem 4 (Luo)**.** *Let $n-1 \geq k \geq 4$. Let G be a connected n-vertex graph with no P_k, then the number of s-cliques in G is at most*

$$\max\{f_s(n,k,\lfloor (k-1)/2 \rfloor), f_s(n,k,1)\}.$$

As a corollary, she also showed

Corollary 1 (Luo)**.** *Let $n \geq k \geq 3$. Assume that G is an n-vertex graph with no cycle of length k or more, then*

$$N_s(G) \leq \frac{n-1}{k-2}\binom{k-1}{s}.$$

Győri, Katona and Lemons [2] initiated the study of Berge P_k-free hypergraphs. They proved

Theorem 5 (Győri-Katona-Lemons)**.** *Let $\mathcal{H}$ be an r-uniform hypergraph with no Berge path of length k. If $k > r+1 > 3$, we have*

$$|E(\mathcal{H})| \leq \frac{n}{k}\binom{k}{r}.$$

If $r \geq k > 2$, we have

$$|E(\mathcal{H})| \leq \frac{n(k-1)}{r+1}.$$

The case when $k = r+1$ was settled later [1]:

Theorem 6 (Davoodi-Győri-Methuku-Tompkins)**.** *Let $\mathcal{H}$ be an n-vertex r-uniform hypergraph. If $|E(\mathcal{H})| > n$, then $\mathcal{H}$ contains a Berge path of length at least $r+1$.*

Our main result is the asymptotic upper bound for the connected version of Theorem 5, as n and k tend to infinity.

Theorem 7. *Let $\mathcal{H}_{n,k}$ be a largest r-uniform connected n-vertex hypergraph with no Berge path of length k, then*

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{|E(\mathcal{H}_{n,k})|}{k^{r-1}n} = \frac{1}{2^{r-1}(r-1)!}.$$

A construction yielding the bound in Theorem 7 is given by partitioning an n-vertex set into two classes A, of size $\lfloor \frac{k-1}{2} \rfloor$, and B, of size $n - \lfloor \frac{k-1}{2} \rfloor$ and taking $X \cup \{y\}$ as a hyperedge for every $(r-1)$-element subset X of A and every element $y \in B$. This hypergraph has no Berge P_k as we could have at most $\lfloor \frac{k-1}{2} \rfloor$ basic vertices in A and $\lfloor \frac{k-1}{2} \rfloor + 1$ basic vertices in B, thus yielding less than the required $k+1$ basic vertices.

Observe that in Theorem 5 the corresponding limiting value of the constant factor is $\frac{1}{r!}$ which is $\frac{2^{r-1}}{r}$ times larger than in the connected case. Note that the ideas of the proof of Theorem 7 can be used to prove that the limiting value of the constant factor in Theorem 5 is $\frac{1}{r!}$.

2 Proof of Theorem 7

We will use the following simple corollary of Theorem 4.

Corollary 2. *4 Let G be a connected graph on n vertices with no P_k, then G has at most*

$$\frac{k^{r-1}n}{2^{r-1}(r-1)!}$$

r-cliques if $n \geq c_{k,r}$ for some constant $c_{k,r}$ depending only on k and r.

Given an r-uniform hypergraph $\mathcal{H}$ we define the shadow graph of $\mathcal{H}$, denoted $\partial\mathcal{H}$ to be the graph on the same vertex set with edge set:

$$E(\partial\mathcal{H}) := \{\{x,y\} : \{x,y\} \subset e \in E(\mathcal{H})\}.$$

Definition 2. *If $r = 3$, then we call an edge $e \in E(\partial\mathcal{H})$* fat *if there are at least 2 distinct hyperedges h_1, h_2 with $e \subset h_1, h_2$. If $r > 3$, then we call an edge $e \in E(\partial\mathcal{H})$* fat *if there are at least k distinct hyperedges $h_1, h_2, \ldots, h_k$ in $\mathcal{H}$ with $e \subset h_i$ for $1 \leq i \leq k$.*
We call an edge $e \in E(\partial\mathcal{H})$ thin *if it is not fat.*

Thus, the set $E(\partial\mathcal{H})$ decomposes into the set of fat edges and the set of thin edges. We will refer to the graph whose edges consist of all fat edges in $\partial\mathcal{H}$ as the *fat graph* and denote it by F.

Due to space constraintes we omit the proofs of the following three lemmas, full proofs can be found in the article [7].

Lemma 1. *There is no P_k in the fat graph F of the hypergraph $\mathcal{H}$.*

We call a hyperedge $h \in E(\mathcal{H})$ *fat* if h contains no thin edge. Let $\mathcal{F}$ denote the hypergraph on the same set of vertices as $\mathcal{H}$ consisting of the fat hyperedges, then

Lemma 2. *If $r = 3$, then*

$$|E(\mathcal{H} \setminus \mathcal{F})| \leq \frac{(k-1)n}{2}.$$

If $r > 3$, then

$$|E(\mathcal{H} \setminus \mathcal{F})| \leq \frac{(k-1)^2 n}{2}.$$

Any hyperedge of $\mathcal{F}$ contains only fat edges, so it corresponds to a unique r-clique in F. This implies the following.

Observation 1. *The number of hyperedges in $E(\mathcal{F})$ is at most the number of r-cliques in the fat graph F.*

Lemma 3. *There are no two disjoint cycles of length at least $k/2+1$ in the fat graph F.*

Assume that F has connected components $C_1, C_2, \ldots, C_t$. Trivially, $N_r(F) = \sum_{i=1}^{t} N_r(C_i)$
If $|V(C_i)| \leq k/2$, then trivially

$$N_r(C_i) \leq \binom{|V(C_i)|}{r} \leq \frac{|V(C_i)|^r}{r!} \leq \frac{k^{r-1}\,|V(C_i)|}{2^{r-1}(r-1)!}.$$

So we can assume $|V(C_i)| \geq k/2$. By Lemma 3, we have that for all but at most one i, C_i does not contain a cycle of length at least $k/2+1$. So by Corollary 1, for all but at most one i, say i_0, we have

$$N_r(C_i) \leq \frac{|V(C_i)|-1}{k/2-2}\binom{k/2-1}{r} \leq \frac{k^{r-1}\,|V(C_i)|}{2^{r-1}(r-1)!} + O(k^{r-2}).$$

If $|V(C_{i_0})| \geq c_{k,r}$, then by Lemma 1 and by Corollary 2 we have

$$N_r(C_{i_0}) \leq \frac{k^{r-1}\,|V(C_i)|}{2^{r-1}(r-1)!}.$$

Otherwise, $N_r(C_{i_0}) \leq \binom{|V(C_{i_0})|}{r} = o(n)$. Therefore, we have

$$N_r(F) = \sum_{i=1}^{t} N_r(C_i) \leq \sum_{i=1}^{t}\left(\frac{k^{r-1}\,|V(C_i)|}{2^{r-1}(r-1)!} + O(k^{r-2})\right) + o(n) \leq \frac{k^{r-1}n}{2^{r-1}(r-1)!} + O(k^{r-2})n + o(n).$$

Therefore, by Observation 1,

$$|E(\mathcal{F})| \leq N_r(F) \leq \frac{k^{r-1}n}{2^{r-1}(r-1)!} + O(k^{r-2})n + o(n). \tag{1}$$

Since $|E(\mathcal{H})| = |E(\mathcal{H}\setminus\mathcal{F})| + |E(\mathcal{F})|$, adding up the upper bounds in (1) and Lemma 2, we obtain the desired upper bound on $|E(\mathcal{H})|$.

□

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