Enumeration of Unsensed Orientable and Non-Orientable Maps

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Abstract

The work is devoted to enumeration of unlabelled maps on orientable or non-orientable surfaces up to all their homeomorphisms. On the first step we classify all periodic orientation-reversing homeomorphisms of orientable genus g surfaces. Then we derive recurrence relations for quotient maps on orbifolds corresponding to such homeomorphisms, which allows us to enumerate reflexive maps on orientable surfaces. Combining these results with the results obtained by Mednykh and Nedela for sensed maps, we enumerate unsensed maps on orientable surfaces of a given genus. Then we apply an analogous approach to enumerate periodic homeomorphisms of non-orientable surfaces with g crosscaps. By counting quotient maps on orbifolds corresponding to such homeomorphisms we obtain the numbers enumerating unsensed maps on non-orientable surfaces.

By a topological map M on a surface S we will mean a 2-cell imbedding of a connected graph G, loops and multiple edges allowed, into a compact connected 2-dimensional manifold S without boundary, such that the connected components of $S - G$ are 2-cells. The 0-, 1-, and 2-dimensional cells of a map M are its vertices, edges, and faces, respectively [1]. We consider in this work maps on orientable or non-orientable surfaces of Euler characteristic χ. Every orientable surface S can be viewed as a sphere with $g = (2-\chi)/2$ handles for some even $\chi \leq 2$; every non-orientable surface S can be viewed as a sphere with $g = 2 - \chi$ crosscaps for some integer $\chi \leq 1$. The invariant g is called the genus of the surface S.

Two topological maps M_1 and M_2 on surfaces S_1 and S_2 are isomorphic if there is a homeomorphism h from the surface S_1 to the surface S_2 that induces an isomorphism of the underlying graphs G_1 and G_2. By a sensed (unsensed) orientable map we mean an equivalence class of isomorphic orientable maps where the equivalence relation is given by sense-preserving (sense-preserving or sense-reversing) map homeomorphisms. By an unsensed non-orientable map we mean an equivalence class of non-orientable isomorphic maps where the equivalence relation is given by all map homeomorphisms.

A general technique for counting sensed planar maps was developed by Liskovets [2] in the early eighties. His approach reduces the enumerating problem for sensed maps on the sphere to counting *quotient maps on orbifolds*, maps on quotients of a surface under a finite group of automorphisms. His ideas were further developed in a series of papers devoted to enumeration of sensed orientable maps [3], hypermaps [4] and one-face regular maps [5]. In all of these papers a geometric approach based on the enumeration of rooted maps on cyclic orbifolds was employed. A similar approach was applied in the article [6] to enumerate unsensed orientable maps on surfaces regardless of genus. The authors of [6] pointed out that the technique developed in their paper is suitable for enumerating unsensed orientable maps on surfaces of a given genus.

In the present work we modify the technique developed in the papers [3], [4] and [6] to make it suitable for enumerating unsensed orientable and non-orientable maps on a surface of a given Euler characteristic χ. In order to do this, we derive formulas for calculating the number of order-preserving epimorphisms and order-and-orientation-preserving epimorphisms from fundamental groups of orbifolds to cyclic groups. Then, using Tutte's approach [7] based on contracting the root edge, we obtain a system of recurrence relations for determining the numbers of quotient maps

on orientable and non-orientable surfaces with or without a boundary, with or without branch points. Such surfaces arise as orbifolds when considering arbitrary symmetries non-orientable surfaces or orientation-reversing symmetries of non-orientable surfaces.

Consider as an example the class of quotient maps on a disc whose root dart (half of an edge) is incident to a vertex on the boundary, but does not lie on the boundary itself. Without loss of generality, we will assume that the root dart is the leftmost dart incident to the root vertex. The number $d^{(0)}_{n,k}$ of such quotient maps with n darts and the root vertex of degree k can be expressed as follows:

$$d^{(0)}_{n,k} = d^{(0)}_{n-1,k-1} + \sum_{k'=0}^{n-k-1} d^{(0)}_{n-2,k-1+k'} + \sum_{n'=0}^{n-2}\sum_{k'=0}^{k-2} d^{(0)}_{n',k'} \cdot s_{n-n'-2,k-k'-2} +$$

$$+ \sum_{n'=0}^{n-2}\sum_{k'=0}^{n-k-1}\sum_{k''=0}^{k'}\sum_{i_1=0}^{1}\sum_{i_2=0}^{1} d^{(i_1)}_{n-2-n',k+k''-1} \cdot d^{(i_2)}_{n',k'-k''}.$$

Here $d^{(1)}_{n,k}$ is the number of quotient maps on a disc with the root dart that belongs to a boundary edge, k other darts incident to the root which all lie in the interior of the disc, and n darts in total; $s_{n,k}$ is the number of maps with n darts and the root vertex of degree k on a sphere.

Enumerating the quotient maps allows us, in its turn, to enumerate orientable and non-orientable maps up to all homeomorphisms. To the best of our knowledge there are no published analytic results on the enumeration of unsensed orientable and non-orientable maps on surfaces of a given Euler characteristic χ. We provide the results of computing the exact numbers of unsensed maps with n edges in the Tables 1 and 2.

$n\backslash g$	1	2	3	4
2	1	0	0	0
3	6	0	0	0
4	40	4	0	0
5	320	76	0	0
6	2946	1395	82	0
7	29364	24950	4348	0
8	309558	427336	160050	7258
9	3365108	6987100	4696504	688976
10	37246245	109761827	118353618	37466297
11	416751008	1668376886	2675297588	1512650776
12	4696232371	24689351504	55758114082	50355225387
13	53186743416	357467967214	1091344752470	1461269893538
14	604690121555	5083309341304	20318440463052	38236656513725

Table 1: Unsensed orientable genus g maps with n edges

$n\backslash g$	1	2	3	4
1	1	0	0	0
2	4	2	0	0
3	19	16	8	0
4	106	137	128	47
5	709	1254	1890	1372
6	5356	12597	27036	31007
7	44558	133518	379491	611322
8	397146	1464725	5229092	11017122
9	3716039	16373700	70805740	186044902
10	35967272	185086459	944106760	2992773591
11	356784008	2106417804	12426068215	46378655568
12	3605014966	24081813881	161793730426	697928466684
13	36955965852	276231542440	2087922762430	10257901689164
14	383320824698	3176840637522	26745380615078	147883893647230

Table 2: Unsensed non-orientable genus g maps with n edges

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