Pebble Exchange on Graphs and Graph Automorphism

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Abstract

Let G be a graph. A graph puzzle $\mathrm{Puz}(G)$ of G is defined as follows. A configuration of $\mathrm{Puz}(G)$ is a bijection from the set of vertices of a board graph G to the set of vertices of a pebble graph G. A move of pebbles is defined as exchanging two pebbles which are adjacent on both a board graph and a pebble graph. For a pair of configurations f and g, we say that f is equivalent to g if f can be transformed into g by a sequence of finite moves.

Let $\mathrm{Aut}(G)$ be the automorphism group of G, and let 1_G be the unit element of $\mathrm{Aut}(G)$. The pebble exchange group of G, denoted by $\mathrm{Peb}(G)$, is defined as the set of all automorphisms f of G such that 1_G and f are equivalent to each other.

In this paper, some basic properties of $\mathrm{Peb}(G)$ are studied. Among other results, it is shown that for any connected graph G, all automorphisms of G are contained in $\mathrm{Peb}(G^2)$, where G^2 is a square graph of G.

1 Introduction

Let G be a finite and undirected graph with no multiple edge or loop. The vertex set of G and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. For two graphs G and H with a common number of vertices, let us consider a puzzle $\mathrm{Puz}(G, H)$, where G is a board graph and H is a pebble graph. We call a bijection f from $V(G)$ to $V(H)$ a *configuration* of $\mathrm{Puz}(G, H)$, and we denote the set of all configurations of $\mathrm{Puz}(G, H)$ by $\mathcal{C}(G, H)$. Given a configuration f, if $f(x) = y$, we consider that the vertex x of the board is occupied by the pebble y.

In $\mathrm{Puz}(G, H)$, two pebbles $y_1 = f(x_1)$ and $y_2 = f(x_2)$ can be exchanged if $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. Then the resultant configuration g satisfies that $g(x_1) = y_2$, $g(x_2) = y_1$ and $g(x) = f(x)$ for any $x \in V(G) \setminus \{x_1, x_2\}$. We call the operation a *move*. If a configuration f is transformed into another configuration g with a finite sequence of moves, we say that f and g are *equivalent*, denoted by $f \sim g$. $\mathrm{Puz}(G, H)$ is called *feasible* if all the configurations of the puzzle are equivalent to each other. Let P_k be the path with k vertices, and let $K_{1,\ell}$ be the star with ℓ pendant vertices.

Example 1. $\mathrm{Puz}(P_4 \times P_4, K_{1,15})$ *corresponds to a well-known puzzle named* 15*-puzzle, by considering the center of* $K_{1,15}$ *a vacant position. For two configurations* $f, g \in \mathcal{C}(P_4 \times P_4, K_{1,15})$*,* f *is equivalent to* g *if and only if* $g^{-1} \circ f$ *is an even permutation on* $V(G)$*([4, 6]).*

We will show some more examples, from Example 2 to Example 5, which are considered as generalizations of Example 1. Suppose that both G and H are bipartite graphs. It is not difficult to see that for $f, g \in \mathcal{C}(G, H)$, if $g^{-1} \circ f$ is an odd permutation, then we have $f \not\sim g$([2]).

Let $\theta(1, 2, 2)$ be a graph such that $V(\theta(1, 2, 2)) = \{v_i : 1 \leq i \leq 7\}$ and $E(\theta(1, 2, 2)) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_7, v_4v_7\}$.

Example 2. *Let* G *be a* 2*-connected non-bipartite graph with* n *vertices. If* G *is not a cycle or* $\theta(1, 2, 2)$*, then* $\mathrm{Puz}(G, K_{1,n-1})$ *is feasible([7]).*

For a positive integer k, a path $P = v_1v_2\cdots v_k$ of a graph G is called a *k-isthmus* if (1) every edge of P is a bridge of G, (2) every vertex of P is a cutvertex of G, and (3) $\deg_G(v_i) = 2$ for $1 < i < k$. For two graphs G and H, the *join* $G + H$ is defined as $V(G + H) = V(G) \cup V(H)$, $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Example 3. *Let $2 \leq k \leq n$. Let G be a connected graph with n vertices except a cycle. Then $\mathrm{Puz}(G, K_k + \overline{K_{n-k}})$ is feasible if and only if G has no k-isthmus([5]).*

Example 4. *Let $2 \leq k \leq n/2$. Let G be a graph with n vertices. Then $\mathrm{Puz}(G, K_{k,n-k})$ is feasible if and only if (1) G is not a cycle, and (2) G is not bipartite, and (3) G has no k-isthmus([1]).*

Example 5. *Let $r \geq 3$ and let $n_1 \leq \ldots \leq n_r$. Let G be a graph with $n = n_1 + n_2 + \cdots + n_r$ vertices. Then $\mathrm{Puz}(G, K_{n_1,n_2,\ldots,n_r})$ is feasible if and only if (1) G is not a cycle, and (2) G has no $(n - n_r)$-isthmus([1]).*

In [2], the above mentioned graph puzzle was formally introduced and some more necessary/sufficient conditions of the feasibility of the puzzle was studied.

In this paper, we will shed light on algebraic property of the puzzle. In the following, we only consider the case where a board graph and a pebble graph are the same, and we denote $\mathcal{C}(G, G)$ and $\mathrm{Puz}(G, G)$ simply by $\mathcal{C}(G)$ and $\mathrm{Puz}(G)$, respectively.

The *automorphism group* of a graph G, denoted by $\mathrm{Aut}(G)$, is the group which consists of all bijections f from $V(G)$ to $V(G)$ such that $f(x_1)f(x_2) \in E(G)$ if and only if $x_1x_2 \in E(G)$. Let 1_G, or simply 1, denote the identity element of $\mathrm{Aut}(G)$. Let us introduce the *pebble exchange group* of G, denoted by $\mathrm{Peb}(G)$, as the group which consists of all automorphisms f of G such that 1_G and f are equivalent in $\mathrm{Puz}(G)$. If $f, g \in \mathrm{Peb}(G)$, there exists a finite sequence of configurations $f = f_0, f_1, \ldots, f_s = g$, where f_i is generated from f_{i-1} by a move of $\mathrm{Puz}(G)$ for all $1 \leq i \leq s$. We remark that for $1 \leq i \leq s - 1$, f_i is not necessarily an automorphism of G. It is not difficult to see that $\mathrm{Peb}(G)$ turns out a normal subgroup of $\mathrm{Aut}(G)$.

In this paper, we will study basic properties of $\mathrm{Peb}(G)$.

2 Main Results

It is known that for any finite group Γ, there exists a graph G such that $\mathrm{Aut}(G) \simeq \Gamma$([3]). By using this fact, we have the following result.

Proposition 1. *For any finite group Γ, there exists a graph G such that $\mathrm{Peb}(G) \simeq \Gamma$.*

Second, we note a simple observation about $\mathrm{Peb}(G)$, where G contains no small cycle. For a graph G, the *girth* of G, denoted by $\mathrm{girth}(G)$, is the order of the smallest cycle contained in G. If G contains no cycle, $\mathrm{girth}(G)$ is defined as ∞.

Proposition 2. *Let G be a connected graph with at least three vertices. If $\mathrm{girth}(G) \geq 5$, then $\mathrm{Peb}(G) \simeq \{1_G\}$.*

Let P_n and C_n denote the path or the cycle graph of order n, respectively. By Proposition 2, we have $\mathrm{Peb}(P_n) \simeq \{1\}$ for $n \geq 3$, and $\mathrm{Peb}(C_n) \simeq \{1\}$ for $n \geq 5$.

The next result is about pebble exchange group of a product of graphs. For two graphs G and H, let $G \times H$ denote a *Cartesian product* of G and H, where $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(u_1, v_1)(u_2, v_2) \in V(G \times H)^2 : u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$.

Theorem 3. *For any two connected graphs G and H,* $\mathrm{Peb}(G \times H) \simeq \mathrm{Peb}(G) \times \mathrm{Peb}(H)$.

Let Q_n be the n-dimensional hypercubic graph. Since $Q_n = P_2^n$ and $\mathrm{Peb}(P_2) \simeq \mathbb{Z}/2\mathbb{Z}$, we have the following corollary as an immediate consequence of Theorem 3.

Corollary 4. *For $n \geq 1$,* $\mathrm{Peb}(Q_n) \simeq (\mathbb{Z}/2\mathbb{Z})^n$.

As a graph G becomes sparse, the number of possible moves of $\mathrm{Puz}(G)$ decreases. Hence, it is interesting to show the existence of graphs G such that $\mathrm{Peb}(G)$ has a rich structure with $|E(G)| = O(|V(G)|)$. For a graph G, the *square graph* G^2 of G is defined as $V(G^2) = V(G)$ and $E(G^2) = \{uv \in V(G)^2 : d_G(u,v) = 1 \text{ or } 2\}$, where $d_G(u,v)$ is the *distance* between u and v in G.

The main result of the paper is the following theorem.

Theorem 5. *For any connected graph G,* $\mathrm{Peb}(G^2) \supset \mathrm{Aut}(G)$.

In the remaining of the paper, we show the sketch of the proof of Theorem 5. First, we deal with the simplest but the most important case, where G is a path.

Lemma 6. *For $n \geq 2$,* $\mathrm{Peb}(P_n^2) \supset \mathrm{Aut}(P_n)$.

Proof. It is not difficult to see that for $n \leq 5$, we have $\mathrm{Puz}(P_n^2)$ is feasible. Hence, we have $\mathrm{Peb}(P_n^2) = \mathrm{Aut}(P_n^2) \supset \mathrm{Aut}(P_n)$. Suppose that $n \geq 6$. In this case, since $\mathrm{Aut}(P_n^2) = \mathrm{Aut}(P_n) \simeq \mathbb{Z}/2\mathbb{Z}$, it suffices to prove $\mathrm{Peb}(P_n^2) \simeq \mathbb{Z}/2\mathbb{Z}$. Let us label the vertices of P_n as $V(P_n) = \{1, 2, \ldots, n\}$ and $E(P_n) = \{ij : j - i = 1\}$. Note that $\mathrm{Aut}(P_n^2) = \{1_n, \alpha_n\}$, where $1_n(i) = i$ for $1 \leq i \leq n$ and $\alpha_n(i) = n - i + 1$ for $1 \leq i \leq n$. It suffices to show that $1_n \sim \alpha_n$ in $\mathrm{Puz}(P_n^2)$.

In the following, besides $\mathrm{Puz}(P_n^2)$, we consider two additional puzzles $\mathrm{Puz}(P_{n+1}^2 \setminus \{n\}, P_n^2)$ and $\mathrm{Puz}(P_n^2, P_{n+1}^2 \setminus \{n\})$. For configurations $f \in \mathcal{C}(P_n^2, P_n^2)$, $g \in \mathcal{C}(P_{n+1}^2 \setminus \{n\}, P_n^2)$ and $h \in \mathcal{C}(P_n^2, P_{n+1}^2 \setminus \{n\})$, we will use notations as

$$\begin{aligned} f &= (f(1), f(2), \ldots, f(n-1), f(n)), \\ g &= (g(1), g(2), \ldots, g(n-1), *, g(n+1)), \\ h &= (h(1), h(2), \ldots, h(n-1), h(n)). \end{aligned}$$

By using this notation, 1_n and α_n is expressed as

$$1_n = (1, 2, \ldots, n-1, n), \qquad \alpha_n = (n, n-1, \ldots, 2, 1).$$

Let us define $1'_n$ and $\beta_n \in \mathcal{C}(P_{n+1}^2 \setminus \{n\}, P_n^2)$ as

$$1'_n = (1, 2, \ldots, n-1, *, n), \qquad \beta_n = (n, n-1, \ldots, 2, *, 1),$$

and let us define $1''_n$ and $\gamma_n \in \mathcal{C}(P_n^2, P_{n+1}^2 \setminus \{n\})$ as

$$1''_n = (1, 2, \ldots, n-1, n+1), \qquad \gamma_n = (n+1, n-1, \ldots, 2, 1).$$

What we want to show is that $1_n \sim \alpha_n$, $1'_n \sim \beta_n$, $1''_n \sim \gamma_n$ for $n \geq 1$.
Note that $P_{n+1}^2 \setminus \{n\}$ is naturally considered as a subgraph of P_n^2. Hence, $1'_n \sim \beta_n$ implies that $1_n \sim \alpha_n$. Furthermore, $\mathrm{Puz}(P_{n+1}^2 \setminus \{n\}, P_n^2)$ and $\mathrm{Puz}(P_n^2, P_{n+1}^2 \setminus \{n\})$ are isomorphic as puzzles, since these puzzles can be switched to each other by interchanging the roles of a board graph and a pebble graph. Hence, $1'_n \sim \beta_n$ holds if and only if $1''_n \sim \gamma_n$ holds.

We proceed by induction on n. For $n \leq 2$, it is not difficult to see that the conclusion holds. Let $n \geq 3$. It suffices to show that $1'_n \sim \beta_n$ by using the inductive assumptions $1_k \sim \alpha_k$, $1'_k \sim \beta_k$, $1''_k \sim \gamma_k$ for $2 \leq k \leq n-1$. By the inductive hypothesis, we have

$$\begin{aligned}
1'_n &= (1, 2, \ldots, n-2, n-1, *, n) \\
&\sim (1, n, \ldots, 4, 3, *, 2) \quad \text{by } 1'_{n-1} \sim \beta_{n-1} \\
&\sim (1, 3, \ldots, n-1, n, *, 2) \quad \text{by } 1_{n-2} \sim \alpha_{n-2} \\
&\sim (n, n-1, \ldots, 3, 1, *, 2) \quad \text{by } 1''_{n-1} \sim \gamma_{n-1} \\
&\sim (n, n-1, \ldots, 3, 2, *, 1) \quad \text{by the exchange of 1 and 2} \\
&= \beta_n,
\end{aligned}$$

as required. □

Second, let us introduce a new operation, *path flip*, for a configuration $f \in \mathcal{C}(G)$. Let $P = v_0 v_1 \ldots v_n$ be a path of G. If $f(v_0)f(v_1)\ldots f(v_n)$ is also a path of G, by a path flip, f can be replaced with $g \in \mathcal{C}(G)$ such that $g(v_i) = f(v_{n-i})$ for $0 \leq i \leq n$, and $g(x) = f(x)$ for all $x \in V(G) \setminus V(P)$.

The following lemma may arouse an independent interest apart from pebble exchange puzzles.

Lemma 7. *For a connected graph G, and for any two configurations f, $g \in \mathrm{Aut}(G)$, f can be transformed into g by a finite sequence of path flips.*

By Lemma 6 and 7, Theorem 5 follows.

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