A general decomposition theory for the 1-2-3 Conjecture and locally irregular decompositions[1]

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Abstract

The 1-2-3 Conjecture is one of the most popular open problems in graph distinguishing labellings. It claims that we can weight the edges of any graph G with no isolated edge with weights in $\{1, 2, 3\}$ in such a way that every two adjacent vertices of G get different sums of their respective incident weights. The 1-2-3 Conjecture has recently been investigated from a decomposition angle, via so-called locally irregular decompositions, which are edge-partitions into locally irregular subgraphs. Through several recent studies, it was shown that this concept is quite related to the 1-2-3 Conjecture. However, the full connection between all these concepts was not clear.

In this work, we propose an approach that generalizes all concepts above, involving coloured weights and sums. As a consequence, we get another interpretation of several existing results related to the 1-2-3 Conjecture. We also come up with new conjectures, to which we give some support.

1 Introduction

Given a k-edge-weighting ω of a graph G, for a positive integer k, the incident sum $\sigma_\omega(u)$ of a vertex u of G, induced by ω, is defined by:

$$\sigma_\omega(u) = \sum_{e \in E(G), e \ni u} \omega(e).$$

If σ_ω is a proper vertex-colouring of G, then w is a *neighbour-sum-distinguishing* (nsd for short) edge-weighting of G. The smallest integer k such that G has an nsd k-edge-weighting (if any) is denoted by $\chi_\Sigma(G)$. For two graphs G and H, we say that G has no isolated H, if G has no component isomorphic to H. In the study of nsd edge-weightings, we obviously have to consider graphs with no isolated K_2. These graphs are referred to as *nice graphs*.

Nsd edge-weightings were introduced in 2004 by Karoński, Łuczak and Thomason in [4] as a weaker version of the irregularity strength. In the same paper, they asked a question which later became the famous 1-2-3 Conjecture:

Question (1-2-3 Conjecture)**.** *Is it true that $\chi_\Sigma(G) \leq 3$, for every nice graph G?*

[1] A full version of the paper is available at https://hal.archives-ouvertes.fr/hal-01690222

Despite many active investigations in the last decade (see [7]), the 1-2-3 Conjecture is still wide open to date. These investigations have been mainly focused on proving the 1-2-3 Conjecture for new classes of nice graphs, proving general constant upper bounds on χ_Σ and studying side aspects of the 1-2-3 Conjecture. Up to now, the best result towards the 1-2-3 Conjecture was given by Kalkowski, Karoński and Pfender in [3], where they proved that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G.

The current work is also related to locally irregular decompositions, which were considered as a decomposition approach towards understanding some aspects behind the 1-2-3 Conjecture. *Locally irregular graphs* are graphs where, for every edge uv, the degrees of u and v are different. Hence locally irregular graphs are exactly the graphs that admit an nsd 1-edge-weighting. A *locally irregular decomposition* of a graph G is a partition of $E(G)$ into classes inducing locally irregular subgraphs. Similarly as for nsd edge-weightings, there exist graphs which do not admit any locally irregular decomposition; but, this time, the class of *exceptional graphs* is much wider (consider, for instance, any path of odd length). An exceptional graph (with respect to locally irregular decompositions) is also called an *exception*, for short. Conversely, a graph that is not exceptional is said *decomposable*. In their first work on this topic [2], Baudon, Bensmail, Przybyło and Woźniak completely characterized all connected exceptions. Namely, connected exceptions include odd-length paths, odd-length cycles, and a family $\mathfrak{T}$ defined recursively from paths and triangles. The smallest integer k such that a decomposable graph G admits a locally irregular k-decomposition is denoted by $\chi'_{irr}(G)$.

Similarly to the 1-2-3 Conjecture, Baudon, Bensmail, Przybyło and Woźniak proposed the following conjecture in 2015:

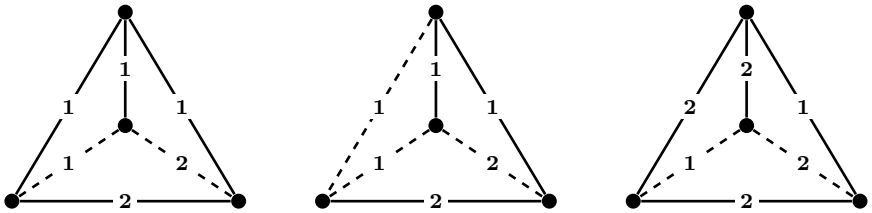
Conjecture 1 ([2])**.** *For every decomposable graph G, $\chi'_{irr}(G) \leq 3$.*

The best known result toward Conjecture 1 is due to Lužar, Przybyło and Soták in [5], where they proved that $\chi'_{irr}(G) \leq 220$ holds for every decomposable graph G, and refined the bound to $\chi'_{irr}(G) \leq 7$ for every decomposable bipartite graph G and to $\chi'_{irr}(G) \leq 4$ for every decomposable subcubic graph G. Meanwhile, Conjecture 1 was verified for several classes of graphs, including decomposable trees (i.e. trees not being an odd-length path), complete graphs on more than 4 vertices and some classes of decomposable bipartite graphs and Cartesian products [2]. Using probabilistic methods, Conjecture 1 has also been verified for regular graphs with degree at least 10^7 [2], and for graphs with minimum degree at least 10^{10} [6].

2 A unified terminology

2.1 Coloured weights and sums

In this work we introduce a new notion of edge-weighting generalising both nsd edge-weightings and locally irregular decompositions. The idea is to assign a colour to each edge in addition of a weight. This allows us to calculate for each vertex a tuple of induced coloured sums. Formally, if $\ell, k \geq 1$ are two integers, and G is a graph, then to each edge of G, we assign, via an edge-colouring ω, a pair (α, β), where $\alpha \in \{1, \ldots, \ell\}$ and $\beta \in \{1, \ldots, k\}$, which can be regarded as a *coloured weight* (with value β and colour α). Now, for every vertex v of G, and every colour $\alpha \in \{1, \ldots, \ell\}$, one can compute the *weighted α-degree* $\sigma_\alpha(v)$ of v, being the sum of weights with colour α incident to v. So, with every vertex v is associated a palette $(\sigma_1(v), \ldots, \sigma_\ell(v))$ of ℓ coloured sums.



(a) Weak. (b) Standard. (c) Strong.

Figure 1: Three $(2,2)$-edge-colourings of K_4.

When dealing with these new notions, there are many ways for asking for distinction, as several coloured sums are available. In this work, we focus on the following three distinction variants. Every two adjacent vertices u and v of G are considered *weakly distinguished* if there is an $\alpha \in \{1, \ldots, \ell\}$ such that $\sigma_\alpha(u) \neq \sigma_\alpha(v)$. Vertices u and v are considered *standardly distinguished* if, assuming $\omega(uv) = (\alpha, \beta)$, we have $\sigma_\alpha(u) \neq \sigma_\alpha(v)$. Finally u and v are considered *strongly distinguished* if, for every $\alpha \in \{1, \ldots, \ell\}$, we have $\sigma_\alpha(u) = \sigma_\alpha(v) = 0$, or $\sigma_\alpha(u) \neq \sigma_\alpha(v)$. Assuming ω verifies one of the weak, standard or strong distinction condition for every pair of adjacent vertices, we say that ω is a *weak*, *standard* or *strong* (ℓ, k)*-edge-colouring*, respectively. We also say that G is *weakly*, *standardly* or *strongly* (ℓ, k)*-edge-coloured*, respectively. Finally, we say that G is *weakly*, *standardly* or *strongly* (ℓ, k)*-edge-colourable*, if there are $\ell', k' \geq 1$ with $\ell' \leq \ell$ and $k' \leq k$ such that G can be weakly, standardly or strongly (ℓ', k')-edge-coloured, respectively.

We provide, in Figure 1, an illustration of these concepts on K_4, the complete graph on 4 vertices, where the two colours are represented by solid and dashed edges. Note that a strong edge-colouring is also a standard edge-colouring which is also a weak edge-colouring. The converse is not always true, however.

2.2 Connecting nsd edge-weightings and locally irregular decompositions

As mentioned earlier, (ℓ, k)-edge-colourings are a generalization of nsd edge-weightings and of locally irregular decompositions. Indeed observe that a weak, standard or strong $(1, k)$-edge-colouring is an nsd edge-weighting, and that a standard $(\ell, 1)$-edge-colouring is a locally irregular decomposition. Moreover, a weak $(\ell, 1)$-edge-colouring is equivalent to a neighbour-multiset-distinguishing ℓ-edge-weighting (where adjacent vertices require distinct multisets of incident weights, see [1]). Using these observations we reformulate the best known results on nsd edge-weightings and locally irregular decompositions in the following theorems:

Theorem 1 ([3])**.** *Every nice graph is strongly* $(1, 5)$*-edge-colourable.*

Theorem 2 ([5])**.** *Every decomposable graph is standardly* $(220, 1)$*-edge-colourable.*

Theorem 3 ([1])**.** *Every nice graph is weakly* $(4, 1)$*-edge-colourable.*

Rephrased differently, the 1-2-3 Conjecture asks whether every nice graph is strongly $(1, 3)$-edge-colourable. Similarly, Conjecture 1 asks whether every decomposable graph is standardly $(3, 1)$-edge-colourable. As a leading objective, we consider the following conjecture, which has flavour of both problems:

Conjecture 2. *Every graph with no isolated edge or triangle is strongly* $(2, 2)$*-edge-colourable.*

3 Our results

While keeping proving Conjecture 2 as our main objective, we also study its standard and weak versions (where the standard and weak distinction conditions are considered). We prove Conjecture 2 for nice complete graphs and bipartite graphs, and we characterize the bipartite graphs that do not admit strong $(1,1)$-, $(1,2)$- or $(2,1)$-edge-colourings, using results proved by Thomassen, Wu and Zhang in [8], which characterize nice bipartite graphs that do not admit an nsd 2-edge-weighting.

Then we explore the standard version of Conjecture 2. We prove it for the following graph families:

Theorem 4. *Every 2-degenerate graph with no isolated edge or triangle is standardly $(2,2)$-edge-colourable.*

Theorem 5. *Every graph with maximum degree 3 with no isolated edge or triangle is standardly $(2,2)$-edge-colourable.*

We also prove constant upper bounds for general graphs and for 9-colourable graphs:

Theorem 6. *Every nice graph G is standardly $(40,3)$-edge-colourable.*

Theorem 7. *Every nice 9-colourable graph G is standardly $(2,3)$-edge-colourable.*

Finally we study the weak version of Conjecture 2, proving more general results:

Theorem 8. *Every nice graph G is weakly $(3,2)$-edge-colourable.*

Theorem 9. *Every nice graph G is weakly $(2,4)$-edge-colourable.*

Theorem 10. *Every graph G with $\delta(G) \geq 59$ is weakly $(2,3)$-edge-colourable.*

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