Turán numbers for theta graphs

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Abstract

The theta graph $\Theta_{\ell,t}$ consists of two vertices joined by t vertex-disjoint paths of length ℓ each. For fixed odd ℓ and large t, we show that the largest graph not containing $\Theta_{\ell,t}$ has at most $c_\ell t^{1-1/\ell} n^{1+1/\ell}$ edges and that this is tight apart from the value of c_ℓ.

Given a graph F, the *Turán number for* F, denoted by $\mathrm{ex}(n, F)$ is the maximum number of edges in an n-vertex graph that contains no subgraph isomorphic to F. Mantel and Turán determined this function exactly when F is a complete graph, and the study of Turán numbers has become a fundamental problem in combinatorics (see [19, 21, 25] for surveys). The Erdős–Stone theorem [12] determines the asymptotic behavior of $\mathrm{ex}(n, F)$ whenever $\chi(F) \geq 3$, and so the most interesting Turán type problems are when the forbidden graph is bipartite.

One of the most well-studied bipartite Turán problems is the even cycle problem: the study of $\mathrm{ex}(n, C_{2\ell})$. Erdős initiated the study of this problem when he needed an upper bound on $\mathrm{ex}(n, C_4)$ in order to prove a theorem in combinatorial number theory [9]. The combination of the upper bounds by Kővari, Sós and Turán [22] and the lower bounds by Brown [3] and Erdős, Rényi and Sós [11] gave the asymptotic formula

$$\mathrm{ex}(n, C_4) \sim \frac{1}{2} n^{3/2}.$$

It is now known that for certain values of n the extremal graphs must come from projective planes [15, 17, 14] and this is conjectured to be the case for all n (see [16]).

A general upper bound of $\mathrm{ex}(n, C_{2\ell})$ of $c_\ell n^{1+1/\ell}$ for sufficiently large n was originally claimed by Erdős [10] and first published by Bondy and Simonovits [2] who showed that one can take $c_\ell = 20\ell$. Subsequent improvements of the best constant c_ℓ to $8(\ell-1)$ by Verstraëte [27], to $(\ell-1)$ by Pikhurko [24], and finally to $80\sqrt{\ell}\log\ell$ by Bukh and Jiang [7] were made, and this final bound is the current record.

As stated above, we have an asymptotic formula for $\mathrm{ex}(n, C_4)$. Additionally, the upper bound on $\mathrm{ex}(n, C_{2\ell})$ is of correct order of magnitude for $\ell \in \{3, 5\}$ [1, 29], i.e., $\mathrm{ex}(n, C_{2\ell}) = \Theta(n^{1+1/\ell})$ for these values of ℓ. However, unlike the case of C_4, the sharp multiplicative constant is not known; see [18] for the best bounds on $\mathrm{ex}(n, C_6)$. The order of magnitude for $\mathrm{ex}(n, C_{2\ell})$ is unknown for any $\ell \notin \{2, 3, 5\}$. The best known general lower bounds are given by Lazebnik, Ustimenko and Woldar [23] (but see [26] for a better bound for the $\mathrm{ex}(n, C_{14})$ case).

Although it is unclear whether or not $\mathrm{ex}(n, C_{2\ell}) = \Omega(n^{1+1/\ell})$ in general, more is known if instead of forbidding a pair of internally disjoint paths of length ℓ between pairs of vertices (that is, a $C_{2\ell}$) one forbids several paths of length ℓ between pairs of vertices. For $t \in \mathbb{N}$, let $\Theta_{\ell,t}$ be the graph made of t internally disjoint paths of length ℓ connecting two endpoints. The study of $\mathrm{ex}(n, \Theta_{\ell,t})$ generalizes the even cycle problem as $\Theta_{\ell,2} = C_{2\ell}$. Faudree and Simonovits showed [13] that

$$\mathrm{ex}(n, \Theta_{\ell,t}) = O_{\ell,t}\left(n^{1+1/\ell}\right).$$

More recently, Conlon showed that this upper bound gives the correct order of magnitude if the number of paths is a large enough constant [8]. That is, there exists a constant c_ℓ such that

$\mathrm{ex}(n, \Theta_{\ell,c_\ell}) = \Theta_\ell(n^{1+1/\ell})$. Verstraëte and Williford [28] constructed graphs with no $\Theta_{4,3}$ that have $(\frac{1}{2} - o(1))n^{5/4}$ edges.

In this paper, we are interested in the behavior of $\mathrm{ex}(n, \Theta_{\ell,t})$ when ℓ is fixed and t is large. When $\ell = 2$, the result of Füredi [20] shows that $\mathrm{ex}(n, \Theta_{2,t}) \sim \frac{1}{2}\sqrt{t}n^{3/2}$. For general ℓ, the result of Faudree and Simonovits gives that $\mathrm{ex}(n, \Theta_{\ell,t}) \le c_\ell t^{\ell^2} n^{1+1/\ell}$. We improve this bound as follows.

Theorem 1. *For fixed $\ell \ge 2$, we have $\mathrm{ex}(n, \Theta_{\ell,t}) \le c_\ell t^{1-1/\ell} n^{1+1/\ell}$.*

When ℓ is odd, we show that the dependence on t in Theorem 1 is correct.

Theorem 2. *Let $\ell \ge 3$ be an odd integer. Then*

$$\mathrm{ex}(n, \Theta_{\ell,t}) = \Omega_\ell\left(t^{1-1/\ell} n^{1+1/\ell}\right).$$

Below we illustrate some of the ideas with a proof of Theorem 1 for the simplest case.

Case $\ell = 3$

As every graph of average degree $4d$ contains a bipartite subgraph of average degree $2d$, and since every graph of average degree d contains a subgraph of minimum degree d, we henceforth assume that the graph is bipartite of minimum degree d.

Lemma 1. *Let r be any vertex of G. Call a vertex u* bad *if $u \ne r$ and u has more than t common neighbors with r. If G is $\Theta_{3,t}$-free, then no neighbor of r is adjacent to t bad vertices.*

Proof. Suppose w is adjacent to bad vertices $u_1, \dots, u_t$. Define a sequence of vertices $z_1, \dots, z_t$ as follows. We let z_i be any common neighbor of r and u_i other than $w, z_1, \dots, z_{i-1}$. It exists since there are more than t common neighbors between r and u_i. Then $(wu_iz_ir)_{i=1}^t$ is a collection of t disjoint paths of length 3 from w to r. □

Proof of Theorem 1 for $\ell = 3$. Let r be any vertex of G. Let $L_0 = \{r\}$. Let L_1 be the set of all the neighbors of r. Let L_2 be the set of all vertices at distance 2 from r that have at most t common neighbors with r. Note that by Lemma 1 each vertex in L_1 has at least $d - t$ neighbors in L_2. Call a vertex $v_1 \in L_1$ a *parent* of $v_2 \in L_2$ if v_1 and v_2 are adjacent. Note that a vertex in L_2 can have at most t parents. Hence, each vertex in L_2 has at least $d - t$ neighbors in $V(G) \setminus L_1$.

Let L_3 be all vertices in $V(G) \setminus L_1$ that are adjacent to some L_2. Call $v_3 \in L_3$ a *descendant* of $v_1 \in L_1$ if there is a path of the form $v_1v_2v_3$ with $v_2 \in L_2$.

Let $B(v_1) \subset L_3$ be all descendants of v_1 that have more than t common neighbors with v_1. By Lemma 1, each $v_2 \in N(v_1)$ has fewer than t neighbors in $B(v_1)$.

Let H be the subgraph of G obtained from G by removing all edges between $B(v_1)$ and $N(v_1)$ for all $v_1 \in L_1$. Since each $v_2 \in L_2$ has at most t parents, each vertex in L_2 has at least $d - t - t(t-1) = d - t^2$ neighbors in L_3.

For a vertex $v_3 \in L_3$, let $p(v_3)$ be the number of paths of the form $rv_1v_2v_3$ with $v_i \in L_i$. We claim that $p(v_3) \le 2t(t-1)$ for every $v_3 \in L_3$. Indeed, suppose the contrary. We will construct a $\Theta_{3,t}$ subgraph as follows. First, we pick any path $rv_1^{(1)}v_2^{(1)}v_3$ counted by $p(u)$. Since v_3 and $v_1^{(1)}$ has at most t common neighbors, and since r and $v_2^{(1)}$ have at most t common neighbors, at most $2t$ paths counted by $p(u)$ intersect $\{v_1^{(1)}, v_2^{(2)}\}$. So, we can pick another path $rv_1^{(2)}v_2^{(2)}v_3$ that is disjoint

from $\{v_1^{(1)}, v_2^{(2)}\}$. We can repeat this, at each step selecting path $rv_1^{(i)}v_2^{(i)}v_3$ that is disjoint from $\bigcup_{j<i}\{v_1^{(j)}, v_2^{(j)}\}$ for $i = 1, \ldots, t$. The paths $rv_1^{(i)}v_2^{(i)}v_3$ together form a $\Theta_{3,t}$. So, $p(u) \leq 2t(t-1)$ after all.

Since each vertex in L_1 has at least $d - t$ neighbors in L_2 and each vertex in L_2 has at least $d - t^2$ neighbors in L_3, it follows that

$$|L_3| \geq \frac{d(d-t)(d-t^2)}{2t(t-1)}.$$

□

Since $|L_3| \leq n$ the result follows.

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