On contact graphs of paths on a grid

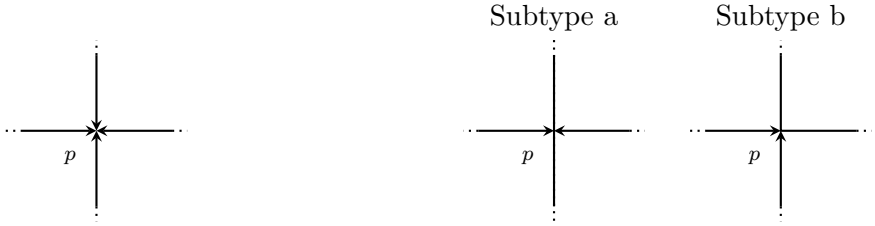
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Abstract

Golumbic et al. introduced in [1] the class of *Vertex intersection graphs of Paths on a Grid* (*VPG graphs*), i.e. graphs for which there exists a collection of nontrivial paths on a grid, referred to as a *representation*, in one-to-one correspondance with their vertex set, such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point. In this paper we consider a natural subclass of VPG graphs, namely that of *Contact graphs of Paths on a Grid* (*CPG graphs*): the class CPG consists of those VPG graphs admitting a representation in which paths are pairwise interiorly disjoint. We examine this class from a structural point of view which leads to constant uppper bounds on the clique number, the chromatic number and the clique chromatic number. We further investigate the relation between planar and CPG graphs and show that CPG graphs are not necessarily planar and not all planar graphs are CPG.

Consider a rectangular grid $\mathcal{G}$ where the horizontal lines are referred to as *rows* and the vertical lines as *columns*. A grid-point lying on row x and column y is denoted by (x, y). If $\mathcal{P}$ is a collection of interiorly disjoint paths on the grid $\mathcal{G}$, the *contact graph of* $\mathcal{P}$ is the graph whose vertex set is in one-to-one correspondance with $\mathcal{P}$, and two vertices are adjacent in the graph if and only if the corresponding paths touch at a grid-point. $\mathcal{R} = (\mathcal{G}, \mathcal{P})$ is then referred to as a *CPG representation of the contact graph of* $\mathcal{P}$, and more specifically a *k-bend CPG representation of the contact graph of* $\mathcal{P}$ if every path of $\mathcal{P}$ has a most k bends (i.e. 90 degrees turns at a grid-point), for $k \geq 0$. A graph G is said to be *CPG* if there exists a collection $\mathcal{P}$ of interiorly disjoint paths on a grid $\mathcal{G}$ such that G is the contact graph of $\mathcal{P}$, and more specifically *B_k-CPG* if every path in $\mathcal{P}$ has at most k bends. In the following, the path representing some vertex u in a CPG representation $\mathcal{R}$ of a graph G is denoted by $P_u^{\mathcal{R}}$, or simply P_u if it is clear from the context.

Let $G = (V, E)$ be a CPG graph and $\mathcal{R} = (\mathcal{G}, \mathcal{P})$ be a CPG representation of G. We say that a grid-point p is of *type I* if it corresponds to an endpoint of four paths in $\mathcal{P}$ (see Fig. 1a), and of *type II* if it corresponds to an endpoint of two paths in $\mathcal{P}$ and an interior point of a third path in $\mathcal{P}$ (see Figure 1b).



(a) Type I. (b) Type II.

Figure 1: Two types of grid-points (the endpoints of a path are marked by an arrow).

For any grid-point p, we denote by $\tau(p)$ the number of edges corresponding to p. For instance, if p is of type I, then $\tau(p) = 6$ since four paths pairwise touching correspond to K_4 which has 6 edges. More generally, $\tau(p) = \binom{k}{2}$ if k paths touch at grid-point p.

For any path P, we denote by $\mathring{P}$ (resp. $\partial(P)$) the interior (resp. endpoints) of P. For a vertex $u \in V$, we define the *weight of u with respect to $\mathcal{R}$*, denoted by $w_u^{\mathcal{R}}$, or simply w_u if it is clear from the context, as follows. Let (x_u^i, y_u^i) $(i = 1, 2)$ be the endpoints of the corresponding path P_u in $\mathcal{P}$ and consider for $i = 1, 2$:

$$w_u^i = |\{P \in \mathcal{P} \mid (x_u^i, y_u^i) \in \mathring{P}\}| + \frac{1}{2} \cdot |\{P \in \mathcal{P} \mid P \neq P_u \text{ and } (x_u^i, y_u^i) \in \partial(P)\}|$$

Then:

$$w_u = w_u^1 + w_u^2$$

Observation 1. *Let $G = (V, E)$ be a CPG graph and $\mathcal{R} = (\mathcal{G}, \mathcal{P})$ be a CPG representation of G. For any vertex $u \in V$ and $i = 1, 2$, $w_u^i \leq \frac{3}{2}$ with equality if and only if (x_u^i, y_u^i) is a grid-point of type I or II.*

Indeed, the contribution of (x_u^i, y_u^i) to w_u^i is maximal if all four grid-edges containing (x_u^i, y_u^i) are used by paths of $\mathcal{P}$, which may only happen when (x_u^i, y_u^i) is a grid-point of type I or II.

Remark. In fact, we have $w_u^i \in \{0, \frac{1}{2}, 1, \frac{3}{2}\}$ for any vertex $u \in V$ and $i = 1, 2$.

Observation 2. *Let $G = (V, E)$ be a CPG graph and $\mathcal{R} = (\mathcal{G}, \mathcal{P})$ be a CPG representation of G. Then,*

$$|E| \leq \sum_{u \in V} w_u$$

with equality if and only if all paths of $\mathcal{P}$ pairwise touch at most once.

Indeed, if $uv \in E$, we may assume, without loss of generality, that either an endpoint of P_u touches the interior of P_v, in which case the edge uv is accounted for in whole in the weight of u, or P_u and P_v have a common endpoint, in which case the edge uv is accounted for in both w_u and w_v by one half.

From these two key observations, we may deduce the following results.

Lemma 1. *A CPG graph is either 6-regular or has a vertex of degree at most 5.*

We can further show that any B_1-CPG graph has a vertex of degree at most 5, a result which no longer holds for B_k-CPG with $k \geq 2$ as there exist 6-regular B_2-CPG graphs (see Fig. 2).

It immediately follows from Observation 2 that CPG graphs cannot contain K_n, for $n \geq 8$. This can further be improved as shown in the next result.

Theorem 1. *CPG graphs are K_7-free.*

Proof. For the sake of contradiction, assume that K_7 is a CPG graph and consider a CPG representation $\mathcal{R} = (\mathcal{G}, \mathcal{P})$ of K_7. First observe that the weight of every vertex with respect to $\mathcal{R}$ must be exactly $2 \cdot 3/2$ (which is equivalent to saying that every grid-point corresponding to an endpoint of a path is either of type I or II) as otherwise the sum of all weights over V would be strictly less than $3|V| = 21 = |E|$, thereby contradicting Observation 2. Similarly, any two paths must touch exactly once, since the sum of all weights over V, equal to 21 as just seen, would otherwise be strictly greater than the number of edges by Observation 2. Hence if we denote by P_I (resp.

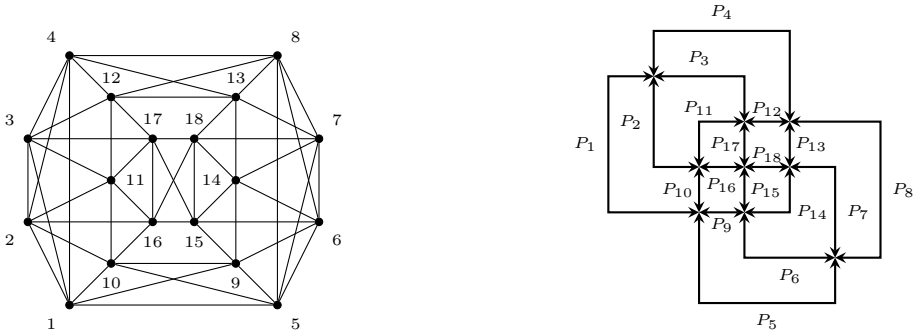


Figure 2: A 6-regular B_2-CPG graph.

P_{II}) the set of grid-points of type I (resp. type II), then since $\tau(p) = 6$ for all $p \in P_I$ and $\tau(p) = 3$ for all $p \in P_{II}$, we have that $6|P_I| + 3|P_{II}| = 21$, i.e. $2|P_I| + |P_{II}| = 7$. Consequently, $|P_{II}| \neq 0$ must be odd, which implies the existence of a path P_u having one endpoint corresponding to a grid-point of type I and the other corresponding to a grid-point of type II. Since the corresponding vertex u has degree 6, P_u must then properly contain an endpoint of another path which, as first observed, necessarily corresponds to a grid-point of type II. But vertex u would then have degree strictly greater than 6 as no two paths touch more than once, a contradiction. □

We can in fact strenghten the latter result for B_1-CPG graphs by showing that K_6 is B_2-CPG but not B_1-CPG. Now by combining Lemma 1 and Theorem 1, we can show that CPG graphs are 6-colorable. Since K_6 is B_2-CPG, this bound is tight for B_k-CPG graphs, with $k \geq 2$. It is not difficult to show that B_0-CPG graphs are 4-colorable (a bound which is tight as K_4 is B_0-CPG), but it remains open whether 6 is a tight bound for B_1-CPG graphs (note that it is at least 5 since K_5 is B_1-CPG). Since any coloring is a clique coloring, it follows that CPG graphs are 6-clique colorable. We can further improve this result in some specific cases.

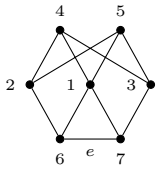
Theorem 2. *Any CPG graph not containing the* 3*-sun as a subgraph is 4-clique colorable.*

Theorem 3. *B_0-CPG graphs are 3-clique colorable.*

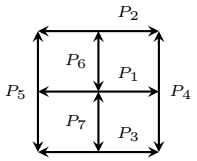
In [3], the authors consider a family of graphs similar to CPG graphs, namely those admitting a *Vertex Contact representation of Paths on a Grid* (*VCPG* for short). Likewise, the vertices of such graphs can be represented by a family of interiorly disjoint paths on a grid, but the adjacencies are defined slightly differently: two vertices are adjacent if and only if the endpoint of one of the corresponding paths touches an interior point of the other corresponding path. It is not difficult to see that graphs admitting a VCPG must be planar, as mentioned in [3], and it immediately follows from the definition that the CPG class contains those graphs. This containment is in fact strict as there exist CPG graphs which are maximally planar and can therefore not admit a VCPG as shown in [3] (see Fig. 4b). We further investigate the relation between planar and CPG graphs.

Lemma 2. *If G is a CPG graph for which there exists a CPG representation containing no grid-point of type I or II.a, then G is planar. In particular, if G is a triangle-free CPG graph, then G is planar.*

It follows from Lemma 2 that CPG graphs are $K_{3,3}$-free. Observe however that for any $k \geq 0$, B_k-CPG is not a subclass of planar graphs as there exist B_0-CPG graphs which are not planar (see Fig. 3).



(a) A nonplanar graph G.



(b) A 0-bend CPG representation of G.

Figure 3: A B_0-CPG graph containing $K_{3,3}$ as a minor (it suffices to contract the edge e).

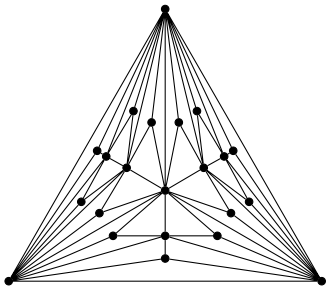
As it was shown in [2] that all triangle-free planar graphs are B_1-CPG, we deduce the following corollary.

Corollary 1. *If a graph G is triangle-free, then G is planar if and only if G is B_1-CPG.*

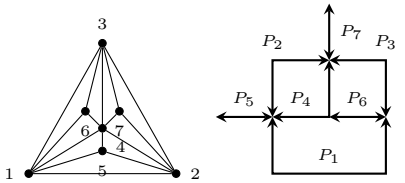
Finally, the next result proves that not all planar graphs are CPG.

Lemma 3. *Let $G = (V, E)$ be a planar graph. If G is a CPG graph, then G has at most $4|V|-2f+4$ vertices of degree at most 3, where f denotes the number of faces of G. In particular, if G is maximally planar, then G has at most 12 vertices of degree at most 3.*

In Figure 4a, we give an example of a maximally planar graph which is not CPG due to Lemma 3. It is constructed by iteratively adding a vertex in a triangular face, starting from the triangle, so that it has exactly 13 vertices of degree 3.



(a) A non CPG maximally planar graph.



(b) A maximally planar CPG graph.

Figure 4: Two maximally planar graphs.

References

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