The second largest eigenvalue and vertex-connectivity of regular multigraphs

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Abstract

Let $\mu_2(G)$ be the second smallest Laplacian eigenvalue of a graph G. The vertex-connectivity of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected. Fiedler proved that $\mu_2(G) \leq \kappa(G)$ for a non-complete simple graph G; for this reason $\mu_2(G)$ is called the "algebraic connectivity" of G. We extend his result to multigraphs. For a pair of vertices u and v, let $m(u, v)$ be the number of edges with endpoints u and v. For a vertex v, let $m(v) = \max_{u \in N(v)} m(v, u)$, where $N(v)$ is the set of neighbors of v, and let $m(G) = \max_{v \in V(G)} m(v)$. We prove that for any multigraph G whose underlying graph is not a complete graph, $\mu_2(G) \leq \kappa(G)m(G)$.

We also prove that for any d-regular multigraph G whose underlying graph is not the complete graph with 2 vertices, if $\mu_2(G) > \frac{d}{4}$, then G is 2-connected. For $t \geq 2$ and infinitely many d, we construct d-regular multigraphs H with $\mu_2(H) = d$, $\kappa(H) = t$, and $m(H) = \frac{d}{t}$. These graphs show that the inequality $\mu_2(G) \leq \kappa(G)m(G)$ is sharp. In addition, we prove that if G is a d-regular multigraph whose underlying graph is not a complete graph, then $\mu_2(G) \leq d$; equality holds for the graphs in the construction.

A *simple* graph is a graph without loops or multiple edges, and a *multigraph* is a graph that may have multiple edges but does not contain loops. In this paper, we treat both a simple graph and a multigraph. The *adjacency matrix* $A(G)$ of G is the n-by-n matrix in which the entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$, where $V(G) = \{v_1, ..., v_n\}$. The *eigenvalues* of G are the eigenvalues of its adjacency matrix $A(G)$. Let $\lambda_i(G)$ be the i-th largest eigenvalue of G. The *Laplacian matrix* of G is $D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. Let $\mu_i(G)$ be the i-th smallest Laplacian eigenvalue of G. Note that $\mu_1(G) = 0$ for any graph G . Also if G is d-regular, then $\lambda_1(G) = d$, and $\lambda_i(G) = d - \mu_i(G)$, thus $\lambda_2(G)$ and $\mu_2(G)$ are directly related.

A multigraph G is *k-vertex-connected* if G has more than k vertices and every subgraph obtained by deleting fewer than k vertices is connected; the *vertex-connectivity* of G, written $\kappa(G)$, is the maximum k such that G is k-vertex-connected. For undefined terms, see West [11] or Godsil and Royle [5].

Fiedler [4] proved that

$$\mu_2(G) \leq \kappa(G) \tag{1}$$

for a non-complete simple graph G; for this reason $\mu_2(G)$ is called the "algebraic connectivity" of G. A lot of research in graph theory over the last 40 years was stimulated by Fiedler's work. We can also easily extend his work to multigraphs with the same idea of his proof for inequality (1). For a pair of vertices u and v, let $m(u, v)$ be the number of edges between u and v. For a vertex v, let $m(v) = \max_{u \in N(v)} m(v, u)$, where $N(v)$ is the set of neighbors of v, and let $m(G) = \max_{v \in V(G)} m(v)$. We call $m(G)$ the multiplicity of the multigraph G. Thoerem 1 is an extension of Fiedler's result for multigraphs.

Theorem 1. *For any multigraph G whose underlying graph is not a complete graph, we have*

$$\mu_2(G) \leq \kappa(G)m(G). \tag{2}$$

Note that Theorem 1 implies Fielder's inequality (1) by taking $m(G) = 1$ (when G is simple). There are graphs showing the tightness of the bound in inequality (2). To prove Theorem 1, we need the following theorem, which is a consequence of the well-known Courant-Fisher Theorem [4].

Theorem 2. *[4] Let* $\mathbf{0}$ *be the null vector and let* $\mathbf{1}$ *be the all 1's column vector. For any multigraph* G*, since* $L(G)\mathbf{1} = 0\mathbf{1}$*, we have*

$$\mu_2(G) = \min_{x \neq \mathbf{0}, x \perp \mathbf{1}} \frac{x^T L(G) x}{x^T x}. \tag{3}$$

A multigraph G is *t-edge-connected* if every subgraph obtained by deleting fewer than t edges is connected; the *edge-connectivity* of G, written $\kappa'(G)$, is the maximum t such that G is t-edge-connected. Since $\kappa'(G) \geq \kappa(G)$ for any multigraph G, we have the following.

Corollary 1. *For any multigraph* G *whose underlying graph is not a complete graph, we have*

$$\mu_2(G) \leq \kappa'(G) m(G). \tag{4}$$

We could not find any multigraph satisfying equality in inequality (4) when $m(G) \geq 2$. However, there exist infinitely many multigraphs whose algebraic connectivity is bigger than their edge-connectivity. Of course, some of them are multigraphs whose underlying graphs are not complete graphs. For example, consider the graph F obtained from the cycle on 4 vertices by duplicating $\frac{d-1}{2}$ times on two incident edges and $\frac{d+1}{2}$ times on the other two incident edges. Then $\mu_2(F) = \frac{3d}{2} - \frac{\sqrt{d^2+8}}{2}$ and $\kappa'(F) = d - 1$, which means $\mu_2(F) > \kappa'(F)$. We believe that there is no big gap between μ_2 and κ' unlike between μ_2 and κ.

Question 1. *Is there a constant* c *such that* $\mu_2(G) \leq \kappa'(G) + c$ *for any multigraph* G*?*

Fiedler's results have been improved and refined by various authors (See [2], [3], [6]). The author ([8]) extended some of these results to multigraphs. A *d-regular multigraph* G is a multigraph such that for any vertex v in $V(G)$, $\sum_{u \in V(G)} m(v,u) = d$, which is the number of edges incident to v.

Theorem 3. *[7] If* G *is a* d*-regular multigraph with* $\lambda_2(G) < \frac{d-1+\sqrt{9d^2-10d+17}}{4}$*, then* $\kappa'(G) \geq 2$*.*

Theorem 4. *[8] For* $t \geq 2$*, if* G *is a* d*-regular multigraph with* $\lambda_2(G) < d - t$*, then* $\kappa'(G) \geq t + 1$*. Furthermore, if* t *is odd and* G *is a* d*-regular multigraph with* $\lambda_2(G) < d - t + 1$*, then* $\kappa'(G) \geq t + 1$*.*

Note that $\lambda_2(G) < \frac{d-1+\sqrt{9d^2-10d+17}}{4}$ in Theorem 3 and $\lambda_2(G) < d - t$ in Theorem 4 can be replaced by $\mu_2(G) > \frac{3d+1-\sqrt{9d^2-10d+17}}{4}$ and $\mu_2(G) > t$, respectively, since $\mu_2(G) = d - \lambda_2(G)$. Also, in Theorem 3 and 4, complete graphs are the exceptions although the author did not mention in the theorems.

The author also characterized when a d-regular multigraph G has $\lambda_2(G) = \theta(d,t)$ and $\kappa'(G) = t$ for every positive integer $t \in [d-1]$, where

$$\theta(d,t) = \begin{cases} \frac{d-1+\sqrt{9d^2-10d+17}}{4} & \text{if } t = 1 \\ d - t & \text{if } t \text{ is even} \\ d - t + 1 & \text{if } t \text{ is odd.} \end{cases}$$

Many researchers investigated relationships between λ_2 (or μ_2) and κ' in a d-regular multigraph as seen above. In this paper, we study a relationship between μ_2 (or λ_2) and κ in a d-regular multigraph. This paper is the first one to contribute to such a relationship, which appears in Theorem 5.

Theorem 5. *If G is a d-regular multigraph with $\mu_2(G) > \frac{d}{4}$, except for the 2-vertex d-regular multigraph, then $\kappa(G) \geq 2$.*

Furthermore, for any fixed positive integer $t \geq 2$ and infinitely many d, there exists a d-regular multigraph $H_{d,t}$ such that $\kappa(H_{d,t}) = t$ and $\mu_2(H_{d,t}) = d$. This means that although μ_2 is fixed like d, the vertex-connectivity of a d-regular graph G might vary. This is a kind of evidence that μ_2 (or λ_2) may not be closely related to the vertex-connectivity while it is deeply related to the edge-connectivity. The existence of $H_{d,t}$ says that the vertex connectivity of a d-regular graph may not influence its algebraic connectivity.

Furthermore, by using the Interlacing Theorem, we observe that for any multigraph G, the $\alpha(G)$-th largest eigenvalue of G is non-negative, where $\alpha(G)$ is the maximum size of a vertex set S such that no pair of vertices in S is adjacent. This fact implies that $\mu_2(G) \leq d$ when G is a d-regular multigraph whose underlying graph is not a complete graph. Graphs $H_{d,t}$ say that this inequality $\mu_2(G) \leq d$ is tight for d-regular multigraphs G regardless of $\kappa(G)$.

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