The class of (P_7, C_4, C_5)-free graphs: decomposition, algorithms, and χ-boundedness

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Abstract

As usual, P_n ($n \geq 1$) denotes the path on n vertices, and C_n ($n \geq 3$) denotes the cycle on n vertices. For a family $\mathcal{H}$ of graphs, we say that a graph G is *$\mathcal{H}$-free* if no induced subgraph of G is isomorphic to any graph in $\mathcal{H}$. We present a decomposition theorem for the class of (P_7, C_4, C_5)-free graphs; in fact, we give a complete structural characterization of (P_7, C_4, C_5)-free graphs that do not admit a clique-cutset. We use this decomposition theorem to show that the class of (P_7, C_4, C_5)-free graphs is χ-bounded by a linear function (more precisely, every (P_7, C_4, C_5)-free graph G satisfies $\chi(G) \leq \frac{3}{2}\omega(G)$). We also use the decomposition theorem to construct an $O(n^3)$ algorithm for the minimum coloring problem, an $O(n^2m)$ algorithm for the maximum weight stable set problem, and an $O(n^3)$ algorithm for the maximum weight clique problem for this class, where n denotes the number of vertices and m the number of edges of the input graph.

The proofs of all our results can be found in [2]. In what follows, all graphs are finite, simple, and nonnull.

Given graphs G and H, we say that G is *H-free* if no induced subgraph of G is isomoprhic to H. Given a graph G and a family $\mathcal{H}$ of graphs, we say that G is *$\mathcal{H}$-free* if G is H-free for all $H \in \mathcal{H}$.

As usual, given a positive integer n, we denote the path on n vertices by P_n, and we denote the complete graph on n vertices by K_n. For an integer $n \geq 3$, C_n is the cycle on n vertices.

A *clique* in a graph G is a (possibly empty) set of pairwise adjacent vertices of G, and a *stable set* in G is a (possibly empty) set of pairwise nonadjacent vertices of G. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G, and the *stability number* of G, denoted by $\alpha(G)$, is the maximum size of a stable set in G. A *q-coloring* of G is a function $c : V(G) \longrightarrow \{1, \ldots, q\}$, such that $c(u) \neq c(v)$ for every edge uv of G. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number q for which there exists a q-coloring of G.

A *clique-cutset* of a graph G is a (possibly empty) clique C of G such that $G \setminus C$ is disconnected. Given a graph G and disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (respectively: *anti-complete*) to Y in G if every vertex of X is adjacent (respectively: nonadjacent) to every vertex of Y. The complement of a graph G is denoted by $\overline{G}$. An *anticomponent* of a graph G is an induced subgraph H of G such that $\overline{H}$ is a component of $\overline{G}$. (Clearly, the vertex sets of the anticomponents of G are pairwise disjoint and complete to each other.) A *trivial* anticomponent of a graph G is one that has just one vertex; a *nontrivial* anticomponent of G is one that has at least two vertices.

Our main result is the following decomposition theorem for (P_7, C_4, C_5)-free graphs (we remark that not all terms that appear in the theorem have been defined yet).

Theorem 1. *Let G be a graph. Then the following are equivalent:*

- *G is a (P_7, C_4, C_5)-free graph that does not admit a clique-cutset;*
- *either G is a complete graph, or G contains exactly one nontrivial anticomponent, and this anticomponent is either a 7-bracelet, a thickened emerald, a lantern, a 6-wreath, or a 6-crown.*

7-Bracelets, thickened emeralds, lanterns, 6-wreaths, and 6-crowns will be defined later. For now, we remark that Theorem 1 in fact gives a full structural description of (P_7, C_4, C_5)-free graphs that do not admit a clique-cutset. We note, however, that this is not quite a full structure theorem for the class of (P_7, C_4, C_5)-free graphs. This is because P_7 admits a clique-cutset, and so the operation of "gluing along a clique," the operation that "reverses" the clique-cutset decomposition, is not class-preserving.

We use Theorem 1, as well as a number of polynomial-time combinatorial algorithms previously obtained by other researchers (including, notably, Tarjan's tools for handling clique-cutsets [13]), to construct an $O(n^3)$ time algorithm for the minimum coloring problem, an $O(n^2m)$ time algorithm for the maximum weight stable set problem, and an $O(n^3)$ time algorithm for the maximum weight clique problem for the class of (P_7, C_4, C_5)-free graphs, where n denotes the number of vertices and m the number of edges of the input graph. We also use Theorem 1 to prove that every (P_7, C_4, C_5)-free graph G satisfies $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$.

Minimum coloring is NP-hard for (C_4, C_5)-free graphs, and even 3-coloring is NP-complete on this class [9]. Huang [8] proved that 4-coloring P_7-free graphs is NP-complete. In [4], the authors show that there is a polynomial-time algorithm for coloring $(4K_1, C_4, C_5)$-free graphs; note that $(4K_1, C_4, C_5)$-free graphs form a proper subclass of the class of (P_7, C_4, C_5)-free class.

The maximum weight stable set problem is NP-hard for (C_4, C_5)-free graphs; its complexity is unknown for P_7-free graphs but it can be solved in polynomial-time for P_5-free graphs [10].

Any C_4-free graph has $O(n^2)$ maximal cliques [1, 5]. Furthermore, if a graph G has K maximal cliques, they can all be found in $O(Kn^3)$ time [11, 14]. Thus, all maximal cliques of a C_4-free graph can be found in $O(n^5)$ time, and it follows that a maximum weight clique of a C_4-free graph can be found in $O(n^5)$ time. As mentioned above, we show that a maximum weight clique of a (P_7, C_4, C_5)-free graph can be found in $O(n^3)$ time.

A class of graphs is *hereditary* if it is closed under isomorphism and induced subgraphs; it is not hard to see that a class $\mathcal{G}$ is hereditary if and only if there exists a family $\mathcal{H}$ such that $\mathcal{G}$ is precisely the class of $\mathcal{H}$-free graphs. A hereditary class $\mathcal{G}$ is *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. χ-Bounded classes were introduced by Gyárfás [7] in the 1980s as a generalization of perfection (a graph G is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$). As mentioned above, we proved that all (P_7, C_4, C_5)-free graphs G satisfy $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$. Gyárfás [7] showed that for all positive integers n, the class of P_n-free graphs is χ-bounded. It is well known that P_4-free graphs are perfect [12]; however, for $n \geq 5$, the best χ-bounding function known for the class of P_n-free graphs is exponential [6]. On the other hand, since there exist graphs of arbitrarily large girth and chromatic number [3], the class of (C_4, C_5)-free graphs is not χ-bounded.

In the remainder of this document, we define 7-bracelets, thickened emeralds, lanterns, 6-wreaths, and 6-crowns (these are graphs that appear in the statement of Theorem 1, but have not yet been defined).

Given graphs G and H, we say that G is *obtained by blowing up each vertex of H to a nonempty clique* provided that there exists a partition $\{X_v\}_{v \in V(H)}$ of $V(G)$ into nonempty cliques such that for all distinct $u, v \in V(H)$, if $uv \in E(H)$, then X_u is complete to X_v in G, and if $uv \notin E(H)$, then X_u is anticomplete to X_v in G. A *thickened emerald* is any graph that can be obtained from the *emerald* (see Figure 1) by blowing up each vertex to a nonempty clique. A *6-crown* is any graph that can be obtained from C_6^3 or C_6^4 (see Figure 2) by blowing up each vertex to a nonempty clique.

Let B be a graph, let $\{A_i\}_{i \in \mathbb{Z}_7}$ be a partition of $V(B)$, and let $i^* \in \mathbb{Z}_7$. We say that B is a *7-bracelet* with *good pair* $(\{A_i\}_{i \in \mathbb{Z}_7}, i^*)$ provided that the following hold:

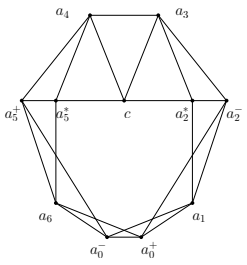


Figure 1: The emerald.

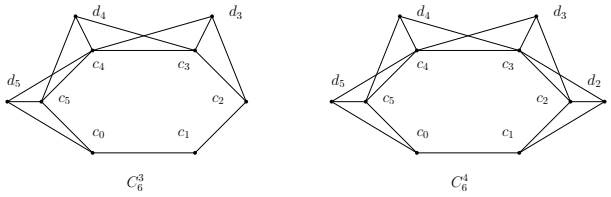


Figure 2: Graphs C_6^3 and C_6^4.

(I) for all $i \in \mathbb{Z}_7$, A_i is a nonempty clique, complete to $A_{i-1} \cup A_{i+1}$ and anticomplete to $A_{i-3} \cup A_{i+3}$;

(II) for all $i \in \mathbb{Z}_7$, there exists a partition of A_i into three (possibly empty) sets, call them A_i^*, A_i^+, A_i^-, such that all the following hold:

- (a) A_i^* is anticomplete to $A_{i-2} \cup A_{i+2}$,
- (b) A_i^+ is anticomplete to A_{i-2}, and every vertex in A_i^+ has a neighbor in A_{i+2},
- (c) A_i^- is anticomplete to A_{i+2}, and every vertex in A_i^- has a neighbor in A_{i-2},
- (d) if $A_i^+ \neq \emptyset$, then A_i^+ can be ordered as $A_i^+ = \{a_1^{i^+}, \ldots, a_{|A_i^+|}^{i^+}\}$ so that $N_B[a_{|A_i^+|}^{i^+}] \subseteq \cdots \subseteq N_B[a_1^{i^+}]$,
- (e) if $A_i^- \neq \emptyset$, then A_i^- can be ordered as $A_i^- = \{a_1^{i^-}, \ldots, a_{|A_i^-|}^{i^-}\}$ so that $N_B[a_{|A_i^-|}^{i^-}] \subseteq \cdots \subseteq N_B[a_1^{i^-}]$,
- (f) either $A_i^* \neq \emptyset$, or A_i^+ is not complete to A_{i+2}, or A_i^- is not complete to A_{i-2},

(III) $A_{i^*-3} = A_{i^*-3}^*$ and $A_{i^*+3} = A_{i^*+3}^*$;

(IV) $A_{i^*-2} = A_{i^*-2}^* \cup A_{i^*-2}^+$ and $A_{i^*+2} = A_{i^*+2}^* \cup A_{i^*+2}^-$;

(V) $A_{i^*-1} = A_{i^*-1}^* \cup A_{i^*-1}^+$ and $A_{i^*+1} = A_{i^*+1}^* \cup A_{i^*+1}^-$.

A *lantern* is a graph R whose vertex set can be partitioned into nonempty cliques A, $B_1, \ldots, B_r$, $C_1, \ldots, C_r$, D (with $r \geq 3$) such that all the following hold:

- A is anticomplete to D;
- A is complete to $\bigcup_{i=1}^r B_i$ and anticomplete to $\bigcup_{i=1}^r C_i$;

- D is complete to $\bigcup_{i=1}^{r} C_i$ and anticomplete to $\bigcup_{i=1}^{r} B_i$;
- B_1 and C_1 can be ordered as $B_1 = \{b_1^1, \ldots, b_{|B_1|}^1\}$ and $C_1 = \{c_1^1, \ldots, c_{|C_1|}^1\}$ so that $N_R[b_{|B_1|}^1] \cap C_1 \subseteq \cdots \subseteq N_R[b_1^1] \cap C_1 = C_1$ and $N_R[c_{|C_1|}^1] \cap B_1 \subseteq \cdots \subseteq N_R[c_1^1] \cap B_1 = B_1$;
- for all $i \in \{2, \ldots, r\}$, B_i is complete to C_i;
- for all distinct $i, j \in \{1, \ldots, r\}$, $B_i \cup C_i$ is anticomplete to $B_j \cup C_j$.

A *6-wreath* is a graph R whose vertex set can be partitioned into six nonempty sets, say $X_0, \ldots, X_5$ (with indices understood to be in $\mathbb{Z}_6$), that can be ordered as $X_0 = \{u_1^0, \ldots, u_{|X_0|}^0\}, \ldots, X_5 = \{u_1^5, \ldots, u_{|X_5|}^5\}$ so that both the following hold:

- for all $i \in \mathbb{Z}_6$, $X_i \subseteq N_R[u_{|X_i|}^i] \subseteq \cdots \subseteq N_R[u_1^i] = X_{i-1} \cup X_i \cup X_{i+1}$;
- X_0 is complete to X_1, X_2 is complete to X_3, and X_4 is complete to X_5.

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