On Characterising and Recognising g_B-Perfect Graphs

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Abstract

The vertex colouring game g_B is played on a simple graph with k permissible colours. Two players, Alice and Bob, take turns to colour an uncoloured vertex such that adjacent vertices receive different colours. Bob makes the first move. The game ends once, the graph is fully coloured, in which case Alice wins, or the graph can no longer be fully coloured, in which case Bob wins. A graph is g_B-perfect if for every induced subgraph H, the game g_B played on H admits a winning strategy for Alice with only $\omega(H)$ colours, where $\omega(H)$ denotes the clique number of H. We characterise g_B-perfect graphs in two ways, by forbidden induced subgraphs and by explicit structural descriptions. We also present a clique module decomposition, allowing us to efficiently recognise g_B-perfect graphs.

1 Introduction

We consider the following maker-breaker graph colouring game g_B played on an initially uncoloured simple graph G with a set of colours C. Bob starts the game by colouring a vertex. Alice and Bob then take turns to apply a colour from C to an uncoloured vertex such that adjacent vertices receive different colours. The game ends once such a move is no longer possible. Alice wins if every vertex is now coloured, otherwise Bob wins. Our game g_B is a variation of Bodlaender's graph colouring game [3], which requires Alice to make the opening move but is otherwise identical.

For any graph G, the *game chromatic number* $\chi_{g_B}(G)$ of G (with respect to the game g_B) denotes the cardinality of a smallest colour set C that admits a winning strategy for Alice at the game g_B. A graph G is called *g_B-perfect* (or *game-perfect* with respect to the game g_B) if, for any induced subgraph H of G, we have $\chi_{g_B}(H) = \omega(H)$, where $\omega(H)$ denotes the clique number of H.

Game-perfect graphs for Bodlaender's original game and two further variants thereof have been characterised by means of forbidden induced subgraphs and by explicit structural characterisations [2]. These results imply a characterisation of non-connected g_B-perfect graphs as the graphs of type $E_1^\cup$ defined below Figure 1. Our main result is to extend this characterisation to the non-trivial case of connected g_B-perfect graphs.

Theorem 1. *Let G be a graph. Then the following are equivalent.*

(i) G is g_B-perfect.

(ii) G contains no induced $F_1, \ldots, F_{15}$ from Figure 2.

(iii) G is an instance of one of the structures $E_1^\cup, E_2, \ldots, E_9$ defined in Figure 1.

The characterisation of g_B-perfect graphs by the explicit structures in Theorem 1 (iii) enables us to efficiently characterise g_B-perfect graphs. For that purpose we use a *clique module decomposition* of a graph as follows. A *module* of a graph $G = (V, E)$ is a set M of vertices such that for every $v \in V \setminus M$ either v is connected to every vertex of M or not connected to any vertex of M. If a module is a clique, it is called *clique module*. A clique module M is *maximal* if there is no clique module $M' \neq M$ with $M \subset M'$.

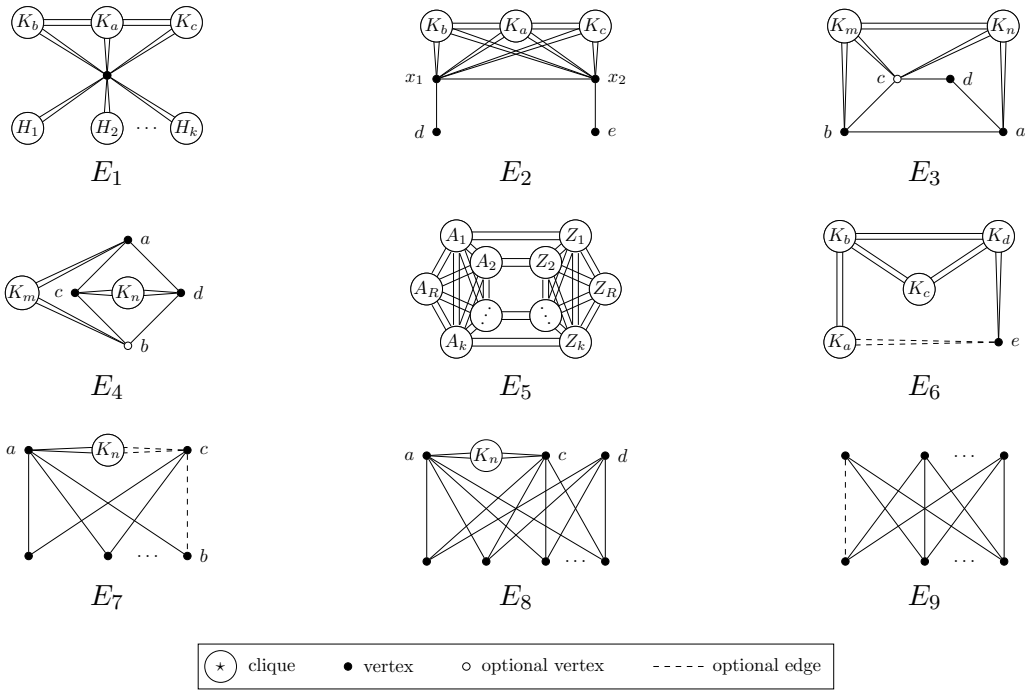


Figure 1: The 9 structural possibilities for connected g_B-perfect graphs: the cliques denoted by large circles have order ≥ 0 in E_1 and ≥ 1 in the other classes. Further we have $k \geq 0$ for E_1 and $k \geq 1$ for E_5. In E_7, vertex c must be completely connected to vertex b or to K_n. We define $E_1^{\cup}$ as the class of disjoint unions of one or several structures of type E_1.

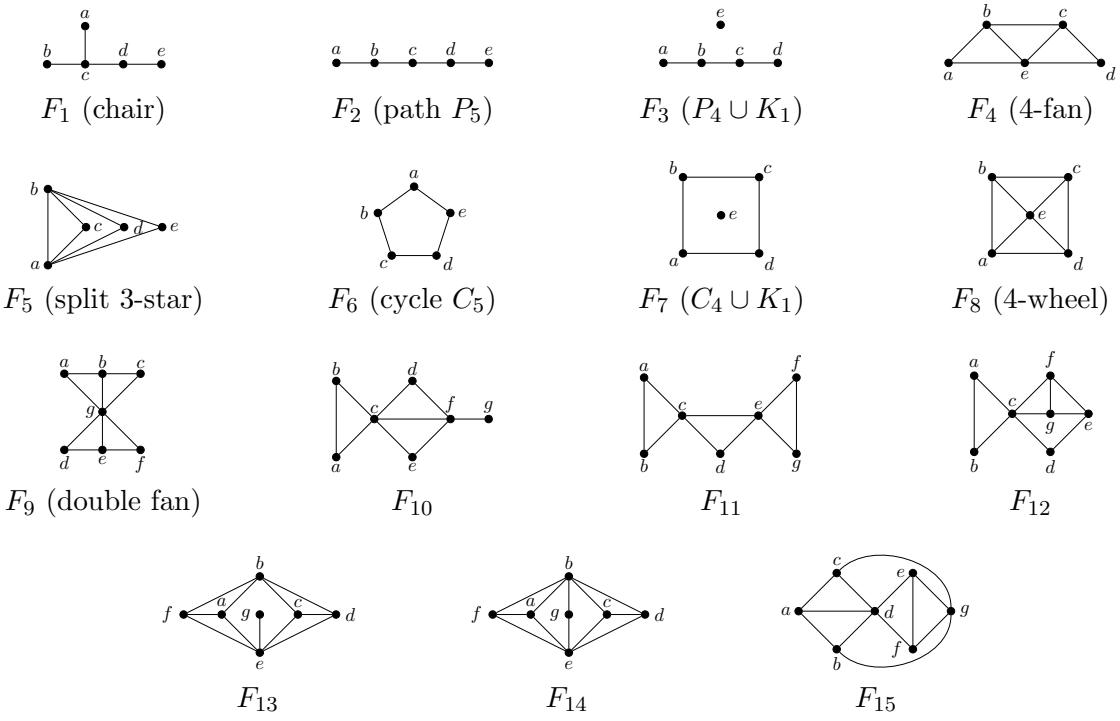


Figure 2: The 15 forbidden induced subgraphs for g_B-perfect graphs

Lemma 1. *Every graph $G = (V, E)$ admits a unique decomposition into maximal clique modules. This decomposition can be determined in time $O(|V|^2)$.*

In Section 3 we describe a simple clique module decomposition algorithm used to prove Lemma 1. More sophisticated general modular decomposition algorithms in time $O(|E|)$ are known [7]. Using Theorem 1 and Lemma 1 we can prove the following.

Theorem 2. *There is a polynomial time algorithm to decide whether a given graph is g_B-perfect.*

g_B-perfect graphs are in particular perfect. The Hamiltonian Cycle problem is NP-complete for perfect graphs, even for bipartite graphs [5]. But for g_B-perfect graphs, by Theorem 2 we obtain

Corollary 1. *The* Hamiltonian Cycle *problem for g_B-perfect graphs is in P.*

2 Proof of Theorem 1

(**i**) $\Longrightarrow$ (**ii**): For every $1 \leq i \leq 15$, we provide a winning strategy for Bob on F_i with $|C| = \omega(F_i)$ colours for the game g_B.

(**iii**) $\Longrightarrow$ (**i**): Let G be an instance of $E_1^{\cup}, E_2, \ldots, E_9$. We first observe that any induced subgraph H of G is also an instance of $E_1^{\cup}$ or E_ℓ for some $2 \leq \ell \leq 9$. Then we provide strategies for Alice to win at the game g_B on any instance H of $E_1^{\cup}, E_2, \ldots, E_9$ with $\omega(H)$ colours.

(**ii**) $\Longrightarrow$ (**iii**): Let $G = (V, E)$ be a connected graph that contains no induced $F_1, \ldots, F_{15}$. We can assume that $|V| \geq 2$. Our aim is to show that G is an instance of one of the classes $E_1, \ldots, E_9$. The following theorem provides the key to our approach. Recall that a *dominating edge* in a graph is an edge e such that every vertex of the graph is adjacent to at least one endpoint of e.

Theorem 3 (Cozzens and Kelleher (1990) [4])**.** *Let H be a connected graph with at least two vertices. If H contains no induced P_5, C_5 or $P_4 \cup K_1$, then it has a dominating edge.*

By Theorem 3 the graph G has a dominating edge x_1x_2. Therefore $V \backslash \{x_1, x_2\}$ can be partitioned into three vertex sets V_1, V_2 and V_3 that contain the vertices not adjacent to x_2, the vertices not adjacent to x_1, and the vertices adjacent to both of x_1 and x_2, respectively.

The remainder of the proof consists of analysing the possible internal structures of the subgraphs G_1, G_2 and G_3 induced by V_1, V_2 and V_3, respectively, and making a series of nested case distinctions to determine the possible structures of adjacencies between G_1, G_2 and G_3.

The internal structures of G_1, G_2 and G_3 are described by the following two lemmas.

Lemma 2. *G_1 and G_2 may have any number of complete components and at most one non-complete component N each. If such an N exists, it takes the shape $N = K_a \vee (K_b \cup K_c)$.*

Lemma 3. *G_3 is of type $K_a \vee (K_b \cup K_c)$ with $a, b, c \geq 0$.*

The results of our nested case distinctions (cf. [6]) are listed below.

G_1 or G_2 empty			E_1
G_1 and G_2 not empty	G_3 non complete		E_2
	G_3 complete	G_1 or G_2 has non complete component	E_3
		G_1 and G_2 have only isolated vertices	E_3, E_6, E_7, E_9
		G_1 has only isolated vertices, G_2 is of type $K_{n_1} \cup \ldots \cup K_{n_k}$, $k \geq 1$, $n_1 \geq 2$	E_3, E_6, E_7, E_8
		G_1 and G_2 are of type $K_{n_1} \cup \ldots \cup K_{n_k}$, $k \geq 1$, $n_1 \geq 2$	E_3, E_4, E_5

3 Proof of Lemma 1

We use the algorithm in Listing 1 which calculates, given a graph $G = (\{1, \ldots, n\}, E)$, its clique module decomposition $\mathcal{F}$. $N[v] := N(v) \cup \{v\}$, where $N(v)$ denotes the set of neighbours of v in G. The algorithm is correct and takes $O(n^2)$ time. It is easy to see that a decomposition of a graph into maximal clique modules is unique.

Listing 1: A simple clique module decomposition algorithm

```
Set F = {{1,...,n}}
For v = 1,...,n:
        For S in F:
                Remove S and add S ∩ N[v] and S \ N[v] to F unless empty.
Return F.
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4 Open questions and further work

$g_{B,A}$- and $g_{A,A}$-perfect graphs are defined by variants of game g_B in which Alice is permitted to miss her turn. Furthermore, in game $g_{A,A}$, Alice may move first instead of Bob. For both games $g_{B,A}$ and $g_{A,A}$, every odd antihole is a minimal forbidden configuration for game-perfect graphs [1]. Apart from the odd antiholes, 13 minimal forbidden configurations are known for $g_{B,A}$-perfect graphs, and 74 for $g_{A,A}$-perfect graphs [6]. It is unknown whether these lists are complete.

Open question 1. *Characterise the $g_{B,A}$- and $g_{A,A}$-perfect graphs, respectively.*

In view of Corollary 1 we might ask the following.

Open question 2. *Identify other problems that are NP-complete on perfect graphs which can be solved efficiently on g_B-perfect graphs.*

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