Recent developments in reconstruction of infinite graphs

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Abstract

Two graphs G and H are hypomorphic if there is a bijection $\varphi\colon V(G) \to V(H)$ such that $G - v \cong H - \varphi(v)$ for each $v \in V(G)$. A graph G is reconstructible if $H \cong G$ for all H hypomorphic to G.

The question whether all locally finite connected graphs are reconstructible had remained open for the past 50 years. In particular, the Harary-Schwenk-Scott Conjecture from 1972 suggests that all locally finite trees are reconstructible. By work of Bondy and Hemminger ('74), Thomassen ('78) and Andreae ('81), every tree with at most countably many ends is reconstructible. Nash-Williams ('87 & '91) proved that locally finite graphs with a finite number ≥ 2 of ends are reconstructible.

In this talk, I aim to sketch our recent construction of a non-reconstructible tree, and I will try to explain how one can modify our tree-construction to obtain non-reconstructible locally finite graphs with one and countably many ends respectively, complementing the results of Nash-Williams.

Two graphs G and H are *hypomorphic* if there exists a bijection φ between their vertex sets such that the induced subgraphs $G - v$ and $H - \varphi(v)$ are isomorphic for each vertex v of G. We say that a graph G is *reconstructible* if $H \cong G$ for every H hypomorphic to G. The *Reconstruction Conjecture*, a famous unsolved problem attributed to Kelly and Ulam, suggests that every finite graph with at least three vertices is reconstructible.

The corresponding reconstruction problem for infinite graphs is false: the countable regular tree T_∞, and two disjoint copies of it, $T_\infty \dot\cup T_\infty$, are easily seen to be hypomorphic, but not isomorphic. This example, however, contains vertices of infinite degree. Regarding locally finite graphs, Harary, Schwenk and Scott showed in '72 that there exists a non-reconstructible locally finite forest. However, they conjectured that the Reconstruction Conjecture should hold for trees.

Conjecture 1 (The Harary-Schwenk-Scott Conjecture)**.** *Every locally finite tree is reconstructible.*

This conjecture has been verified in a number of special cases. Kelly (in '57) showed that finite trees on at least three vertices are reconstructible. Bondy and Hemminger ('74) showed that every tree with at least two but a finite number of ends is reconstructible, and Thomassen ('78) showed that this also holds for one-ended trees. Andreae ('81) proved that also every tree with countably many ends is reconstructible.

A survey of Nash-Williams on the subject of reconstruction problems in infinite graphs from '91 gave the following three main open problems in this area, which have remained open until now.

Problem 2 (Nash-Williams)**.** *Is every locally finite connected infinite graph reconstructible?*

Problem 3 (Nash-Williams)**.** *If two infinite trees are hypomorphic, are they also isomorphic?*

Problem 4 (Halin)**.** *If G and H are hypomorphic, do there exist embeddings $G \hookrightarrow H$ and $H \hookrightarrow G$?*

A positive answer to Problem 2 or 3 would verify the Harary-Schwenk-Scott Conjecture. Recently, in 2016, we constructed a pair of trees which are not only a counterexample to the Harary-Schwenk-Scott Conjecture, but also answer the three questions of Nash-Williams and Halin in the negative. Our counterexample in fact has bounded degree.

Theorem 5 (Bowler, Erde, Heinig, Lehner, Pitz [1]). *There are two (vertex)-hypomorphic infinite trees T and S with maximum degree three such that there is no embedding $T \hookrightarrow S$ or $S \hookrightarrow T$.*

Our example also provides a strong answer to a question by Andreae ('82) about edge-reconstructibility. Two graphs G and H are *edge-hypomorphic* if there exists a bijection $\varphi\colon E(G) \to E(H)$ such that $G - e \cong H - \varphi(e)$ for each $e \in E(G)$. A graph G is *edge-reconstructible* if $H \cong G$ for all H edge-hypomorphic to G. In '82 Andreae constructed countable forests which are not edge-reconstructible, but conjectured that no locally finite such examples can exist.

Problem 6 (Andreae). *Is every locally finite graph with infinitely many edges edge-reconstructible?*

Our example answers Problem 6 in the negative: the trees T and S we construct for Theorem 5 are also be edge-hypomorphic. Besides answering Problem 6, this appears to be the first known example of two non-isomorphic graphs that are *simultaneously* vertex- and edge-hypomorphic.

The Reconstruction Conjecture has also been considered for general locally finite graphs. Nash-Williams showed in '87 that if $p \geq 3$ is an integer, then any locally finite graph with exactly p ends is reconstructible; and four years later, he showed the same is true for $p = 2$. The case $p = 2$ is significantly more difficult. Broadly speaking this is because every graph with $p \geq 3$ ends has some identifiable finite 'centre', from which the ends can be thought of as branching out. A two-ended graph however can be structured like a double ray, without an identifiable 'centre'.

The case of 1-ended graphs is even harder, and the following problems from Nash-Williams' survey, which would generalise the corresponding results established for trees, have remained open.

Problem 7 (Nash-Williams). *Is every locally finite graph with exactly one end reconstructible?*

Problem 8 (Nash-Williams). *Is every locally finite graph with countably many ends reconstructible?*

Recently, we adapted our methods leading to the above Theorem 5 to also construct examples showing that both of Nash-Williams' questions have negative answers.

Theorem 9 (Bowler, Erde, Heinig, Lehner, Pitz [2]). *There is a connected one-ended non-reconstructible graph with bounded maximum degree.*

Theorem 10 (Bowler, Erde, Heinig, Lehner, Pitz [2]). *There is a connected countably-ended non-reconstructible graph with bounded maximum degree.*

In this talk, I aim to describe the constructions and ideas that let to our examples in Theorems 5, 9 and 10.

References

[1] N. Bowler, J. Erde, P. Heinig, F. Lehner and M. Pitz. A counterexample to the reconstruction conjecture for locally finite trees. Bull. London Math. Soc. 49 (2017), 630-648.

[2] N. Bowler, J. Erde, P. Heinig, F. Lehner and M. Pitz. Non-reconstructible locally finite graphs. Submitted to J. Comb. Ser. B.