# An asymptotic bound for the strong chromatic number

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**Abstract**

The strong chromatic number $\chi_s(G)$ of a graph $G$ on $n$ vertices is the least number $r$ with the following property: after adding $r\lceil n/r\rceil - n$ isolated vertices to $G$ and taking the union with any collection of spanning disjoint copies of $K_r$ in the same vertex set, the resulting graph has a proper vertex-colouring with $r$ colours. We show that for every $c > 0$ and every graph $G$ on $n$ vertices with $\Delta(G) > cn$, $\chi_s(G) \leq (2 + o(1))\Delta(G)$, which is asymptotically best possible.

## 1 Introduction

Let $r$ be a positive integer. Let $G$ be a graph on $n$ vertices, where $r$ divides $n$. We say that $G$ is *strongly $r$-colourable* if it can be properly $r$-coloured after taking the union of $G$ with any collection of spanning disjoint copies of $K_r$ in the same vertex set. Equivalently, $G$ is strongly $r$-colourable if for every partition $\{V_1, \dots, V_k\}$ of $V(G)$ with classes of size $r$, there is a proper vertex colouring of $G$ using $r$ colours with the additional property that every $V_i$ receives all of the $r$ colours. If $r$ does not divide $n$, we say that $G$ is *strongly $r$-colourable* if the graph obtained by adding $r\lceil n/r\rceil - n$ isolated vertices to $G$ is $r$-strongly colourable. The *strong chromatic number* $\chi_s(G)$ of $G$ is the minimum $r$ such that $G$ is $r$-strongly colourable. This notion was introduced independently by Alon [2] and Fellows [5].

### 1.1 Strong chromatic number versus maximum degree

It is an open problem to find the best bound on $\chi_s(G)$ in terms of $\Delta(G)$. Alon [3] proved that $\chi_s(G) \leq c\Delta(G)$ for some constant $c > 0$. Haxell [7] showed that $c = 3$ suffices and later [8] that $c \leq 11/4 + \varepsilon$ suffices given $\Delta(G)$ is large enough with respect to $\varepsilon$.

On the other hand, there are examples showing $c \geq 2$ is necessary. For any $\Delta \geq 1$, consider the graph $G$ on $A_1 \cup B_1 \cup A_2 \cup B_2$ where all these sets are pairwise disjoint, $|A_i| = \Delta$ and $|B_i| = \Delta - 1$ for all $i \in \{1, 2\}$, and add every possible edge between $A_1$ and $A_2$. Then $\Delta(G) = \Delta$, but the partition $\{A_1 \cup B_1, A_2 \cup B_2\}$ shows that $G$ is not strongly $(2\Delta - 1)$-colourable (see also, e.g., [4]).

It is conjectured (first explicitly stated by Aharoni, Berger and Ziv [1, Conjecture 5.4]) that this lower bound is also tight.

**Conjecture 1.** *For every graph $G$, $\chi_s(G) \leq 2\Delta(G)$.*

Conjecture 1 is known to be true for graphs $G$ on $n$ vertices with $\Delta(G) \geq n/6$, proven by Axenovich and Martin [4] and independently by Johansson, Johansson and Markström [10].

A fractional version of Conjecture 1 was proven by Aharoni, Berger and Ziv [1]. We say that a graph on $n$ vertices is *fractionally strongly $r$-colourable* if after adding $r\lceil n/r\rceil - n$ isolated vertices and taking the union with any collection of disjoint spanning copies of $K_r$ in the same vertex set, the graph is fractionally $r$-colourable.

**Theorem 2** (Aharoni, Berger and Ziv [1])**.** *Every graph $G$ is fractionally strongly $r$-colourable, for every $r \geq 2\Delta(G)$.*

## 2 Main results

We prove that Conjecture 1 is asymptotically true if $\Delta(G)$ is linear in $|G|$.

**Theorem 3** (Lo and S. [12])**.** *For all $c, \varepsilon > 0$, there exists $n_0 = n_0(c, \varepsilon)$ such that the following holds: if $G$ is a graph on $n \geq n_0$ vertices with $\Delta(G) \geq cn$, then $\chi_s(G) \leq (2+\varepsilon)\Delta(G)$.*

Given a graph $G$ and a partition $\mathcal{P} = \{V_1, \ldots, V_k\}$ of $V(G)$, we make the following definitions. A subset $S \subseteq V(G)$ is *$\mathcal{P}$-legal* if $|S \cap V_i| \leq 1$ for every $i \in [k]$. A *transversal of $\mathcal{P}$* is a $\mathcal{P}$-legal set of cardinality $|\mathcal{P}|$. An *independent transversal of $\mathcal{P}$* is a transversal of $\mathcal{P}$ which is also an independent set in $G$. We will write *transversal* and *independent transversal* if $G$ and $\mathcal{P}$ are clear from the context.

To prove Theorem 3 it suffices to show that given any partition $\mathcal{P}$ of $V(G)$ with classes of size $r \geq (2+\varepsilon)\Delta(G)$, $V(G)$ can be partitioned into independent transversals of $\mathcal{P}$. Moreover, since Conjecture 1 is known to be true for graphs on $n$ vertices with $\Delta(G) \geq n/6$, we might restrict ourselves to study graphs with $\Delta(G) \leq n/6$, and in such graphs any partition $\mathcal{P}$ of $V(G)$ with parts of size $r = (2+\varepsilon)\Delta(G) < 3\Delta(G)$ will have at least 3 classes. Thus Theorem 3 is implied by the following theorem.

**Theorem 4.** *For all integers $k \geq 3$ and $\varepsilon > 0$, there exists $r_0 = r_0(k, \varepsilon)$ such that the following holds for all $r \geq r_0$: if $G$ is a graph and $\mathcal{P}$ is a partition of $V(G)$ with $k$ classes of size $r \geq (2+\varepsilon)\Delta(G)$, then there exists a partition of $V(G)$ into independent transversals of $\mathcal{P}$.*

By considering the complement graph, Theorem 4 easily yields the following corollary. A *perfect $K_k$-tiling* of a graph $G$ is a spanning subgraph of $G$ with components which are complete graphs on $k$ vertices.

**Corollary 5.** *For all integers $k \geq 3$ and $\varepsilon > 0$, there exists $n_0 = n_0(k, \varepsilon)$ such that the following holds: if $n \geq n_0$ and $G$ is a $k$-partite graph with classes of size $n$ and $\delta(G) \geq (k - 3/2 + \varepsilon)n$, then $G$ has a perfect $K_k$-tiling.*

## 3 Proof sketch

Theorem 3 and Corollary 5 are implied by Theorem 4. To prove Theorem 4 we use the *absorption method*, first used in a systematic way by Rödl, Ruciński and Szemerédi [13] (although similar ideas were used previously, e.g. by Krivelevich [11]). This method has been used since to tackle a lot of problems in extremal combinatorics regarding the existence of spanning graphs or hypergraphs.

To prove Theorem 4, the absorbing method works in two steps: finding an absorbing set and finding an almost-perfect partial strong colouring. These can be summarised in the following way.

1. **Finding an absorbing set**. Given a partition $\mathcal{P}$ we find a small vertex set $A \subseteq V(G)$ which is balanced (i.e. it intersects each class of $\mathcal{P}$ in the same number of vertices) with the property that for every small balanced set $S \subseteq V(G)$, $A \cup S$ can be partitioned into independent transversals.

2. **Finding an almost-perfect partial strong colouring** We find a collection of disjoint independent transversals of $\mathcal{P}$ covering almost all vertices.

Given that we can find an absorbing set and an almost-perfect partial strong colouring, the proof of Theorem 4 can be sketched as follows. First, find a balanced absorbing set $A \subseteq V(G)$ of small size. Remove it to get the graph $G \setminus A$, which has a natural induced partition $\mathcal{P}'$. For this partition, find an almost-perfect partial strong colouring, i.e. a collection $\mathcal{T}'$ of disjoint independent transversals of $\mathcal{P}'$, which covers almost all vertices of $G \setminus A$, except for a set $S$ which is balanced and small. Then, by the property of the absorbing set $A$, there exists a collection $\mathcal{T}''$ of disjoint independent transversals which partitions $A \cup S$. Then $\mathcal{T} = \mathcal{T}' \cup \mathcal{T}''$ is a collection of disjoint independent transversals which covers the whole graph, as desired.

In the following subsections we sketch how to find an absorbing set and an almost-perfect partial strong colouring.

### 3.1 Existence of an absorbing set

Given a graph $G$ and a vertex partition $\mathcal{P} = \{V_1, \dots, V_k\}$, a set $S$ is *$\mathcal{P}$-balanced* if $|S \cap V_i| = |S \cap V_j|$ for all $i, j \in [k]$; or just *balanced* if there is no possible ambiguity. This is the absorbing lemma we prove.

**Lemma 6** (Absorbing lemma)**.** *For all integers $k \geq 3$ and $\varepsilon > 0$, there exist $0 < \beta \ll \alpha \ll \varepsilon$ and $r_0 = r_0(\varepsilon, k)$ such that the following holds for all $r \geq r_0$. Let $G$ be a graph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of $V(G)$ with classes of size $r \geq (2+\varepsilon)\Delta(G)$. Then there exists a $\mathcal{P}$-balanced set $A \subseteq V(G)$ of size at most $\alpha n$ such that for every $\mathcal{P}$-balanced set $S \subseteq V(G)$ of size at most $\beta n$, $A \cup S$ can be partitioned in independent transversals of $\mathcal{P}$.*

We sketch the proof of this lemma. To prove this, we proceed in three steps. First, using a lemma of Haxell [9], we can show the following: for every two vertices $x_i, x_i'$ belonging to the same class $V_i$, there is a set $T$ such that both $T \cup \{x_i\}$ and $T \cup \{x_i'\}$ are independent transversals. That is, every two vertices in the same class $V_i$ are contained in two independent transversals which differ only in $V_i$. Using this lemma and applying the standard 'supersaturation' trick, we can show that for every pair not only one set $T$ as before exists, but $\Omega(r^{k-1})$ of them.

Secondly, using the previous results, for each transversal $S$ (not necessarily independent) we construct 'many' balanced sets $A$ of size $k^2$ such that $A \cup S$ can be partitioned into independent transversals. We call this an *absorbing set for $S$*. Finally, form a set $A$ by selecting every possible balanced set of size $k^2$ independently at random. Almost surely, for every transversal $S$ it will contain 'many' absorbing sets for $S$. So for a small balanced set $S \subseteq V(G)$, we can partition it into transversals and then select for each one of them disjoint absorbing sets in $A$. This gives the desired partition of $A \cup S$ into independent transversals.

### 3.2 Existence of an almost-perfect partial strong colouring

Let $G$ be a graph and $\mathcal{P} = \{V_1, \dots, V_k\}$ a partition of $V(G)$ with classes of size $r$. A *$t$-partial strong colouring* of $G$ with respect to $\mathcal{P}$ is a collection of $t$ disjoint independent transversals of $\mathcal{P}$. If $\chi_s(G) = r$, then $G$ has a $r$-partial strong colouring with respect to $\mathcal{P}$. We show the existence of $(1-\delta)r$-partial strong colourings of $\mathcal{P}$.

**Lemma 7.** *For all integers $k \geq 3$ and $\delta, \varepsilon > 0$, there exists $r_0 = r_0(k, \delta, \varepsilon)$ such that the following holds for all $r \geq r_0$: Let $G$ be a graph and $\mathcal{P}$ be a partition of $V(G)$ with $k$ classes of size $r \geq (2+\varepsilon)\Delta(G)$. Then there exists a $(1-\delta)r$-partial strong colouring of $G$ with respect to $\mathcal{P}$.*

To see this, it is convenient to work in the $k$-uniform hypergraph $H$ given by the independent transversals of $G$. First we use Theorem 2 which implies that given the conditions of Lemma 7, there always exists a perfect fractional strong colouring. In terms of the hypergraph $H$, this is equivalent to the existence of a perfect fractional matching in $H$, i.e. an assignment of non-negative weights to $E(H)$ such that for every vertex $v \in V(H)$, the sum of weights of every edge incident with $v$ is equal to 1.

Now, we select a random subgraph $H' \subseteq H$ guided by the perfect fractional matching. Using concentration inequalities we can show that $H'$ can be selected such that every vertex $x \in V(G)$ is contained in almost the same number of edges of $H'$ and every pair $x, y \in V(G)$ is contained in a bounded number of edges of $H'$. It turns out these two conditions are enough to apply a result of Frankl and Rödl [6] which guarantees the existence of a matching $M$ in $H'$ (i.e. a collection of disjoint edges of $H'$) covering almost all vertices. We are done since we can select $M \subseteq H' \subseteq H$ as our desired collection of independent transversals covering almost all vertices.

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