Spanning trees without adjacent vertices of degree 2

Kasper Szabo Lyngsie — **Technical University of Denmark**
Martin Merker — **Technical University of Denmark**

Abstract

Albertson, Berman, Hutchinson, and Thomassen showed in 1990 that for every natural number k there exists a k-connected graph of which every spanning tree contains vertices of degree 2. Using a result of Alon and Wormald, we show that there exists a natural number d such that every graph of minimum degree at least d contains a spanning tree without adjacent vertices of degree 2. Moreover, we prove that every graph with minimum degree at least 3 has a spanning tree without three consecutive vertices of degree 2.

1 Introduction

This is a shortened version of [5] to which we refer the reader for the proofs and further details. All graphs in this paper are simple and finite unless stated otherwise. A *homeomorphically irreducible tree*, or simply a *HIT*, is a tree without vertices of degree 2. HITs have been enumerated by Harary and Prins [3] in 1959. A homeomorphically irreducible spanning tree of a graph is called a *HIST*. The existence of HISTs in graphs with certain structures was studied by Hill [4] in 1974. Hill conjectured that any triangulation of the plane with at least 4 vertices contains a HIST. Malkevitch [6] made the even stronger conjecture that this also holds for all near-triangulations of the plane. Albertson, Berman, Hutchinson and Thomassen [1] proved Malkevitch's conjecture in 1990. Albertson et al. [1] also showed that for some constant c, any connected graph with n vertices and minimum degree at least $c\sqrt{n}$ contains a HIST. In contrast to this, they also constructed a k-connected graph with no HIST for every natural number k.

In this paper we take a different approach and study a natural relaxation of the notion of homeomorphically irreducible spanning trees. We show that large constant minimum degree is sufficient for the existence of a spanning tree which is not far away from being homeomorphically irreducible. To be more precise, we construct spanning trees which can be obtained from HITs by subdividing each edge at most once. In other words, the vertices of degree 2 in the spanning tree form an independent set.

Theorem 1. *There exists a natural number d such that every graph with minimum degree at least d has a spanning tree T without adjacent vertices of degree 2.*

Figure 1 shows a cubic graph in which any spanning tree contains adjacent vertices of degree 2, thus the number d in Theorem 1 has to be at least 4. Theorem 2 shows however, that minimum degree 3 is sufficient for the existence of a spanning tree with no path of length at least 2 consisting of vertices of degree 2.

Theorem 2. *Every graph with minimum degree at least 3 contains a spanning tree T without three consecutive vertices of degree 2.*

2 Spanning trees without adjacent vertices of degree 2

The main tool in our proof of Theorem 1 is a theorem on the existence of large star-covers in graphs of large minimum degree. A *star* is a tree with at most one vertex of degree greater than 1. A

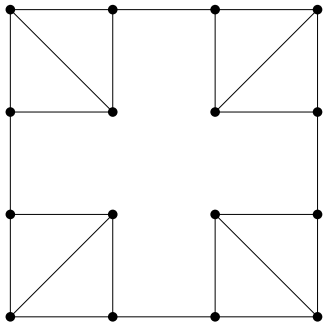


Figure 1: A cubic graph where every spanning tree has two consecutive vertices of degree 2.

star-cover of a graph G is a collection of vertex-disjoint stars in G covering all vertices of G. The following theorem was proved by Alon and Wormald [2].

Theorem 3 (Alon, Wormald [2])**.** *For every natural number d, there exists a natural number $f(d)$ such that every graph of minimum degree at least $f(d)$ has a star-cover where every star has at least d edges.*

Given a star-cover S of a graph G in which every star has large size, every spanning tree T of G containing S does not contain three consecutive vertices of degree 2. An immediate consequence is a weaker version of Theorem 2 where the minimum degree 3 is replaced by a large constant. It seems plausible that by being more careful in the construction of T, one might obtain a spanning tree with even stronger properties. Alon and Wormald [2] asked whether any graph with minimum degree d has a spanning tree where all non-leaves have degree at least $cd/\log d$. However, Albertson et al. [1] showed that large constant minimum degree does not even imply the existence of spanning trees without vertices of degree 2. Based on their examples, we construct for every natural number k a series of graphs with arbitrarily large minimum degree where each spanning tree contains a vertex of degree i for every $i \in \{1, \dots, k\}$. This shows that large minimum degree is not strong enough to avoid any specific degree in a spanning tree. Theorem 1 shows that we can nevertheless obtain a spanning tree which is not too far away from being a HIST, in the sense that it can be obtained from a HIT by subdividing each edge at most once. The strategy for proving Theorem 1 is to first apply Theorem 3 to obtain a star-cover with stars of size at least 6. Given such a star-cover, we grow a tree T by starting with one of the stars and repeatedly adding stars together with some edges to T (and possibly removing some edges), so that T is always a tree without adjacent vertices of degree 2. To make sure that we do not create new vertices of degree 2 when we remove edges, we also require the vertices in T which are adjacent to leaves with neighbours in $G - T$ to have degree at least 5 in T. We use this construction to prove the following lemma which together with Theorem 3 implies Theorem 1.

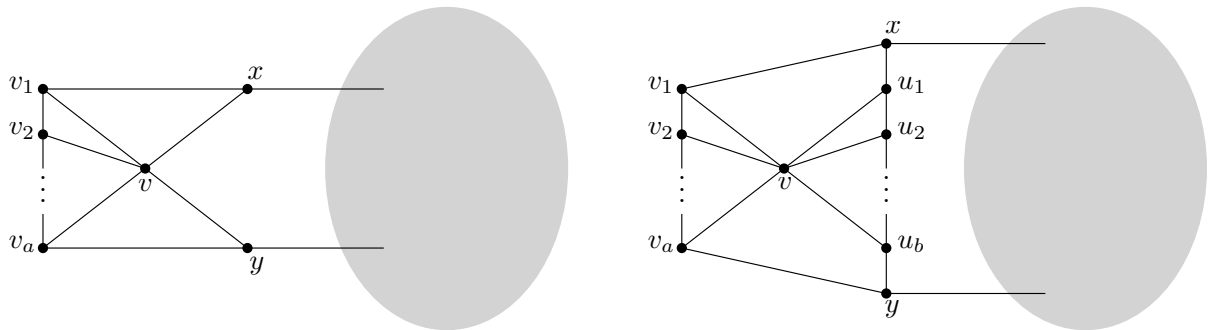
Lemma 1. *Let G be a triangle-free graph of minimum degree at least 3. If G has a star-cover with stars of size at least 6, then G has a spanning tree without adjacent vertices of degree 2.*

3 Spanning trees without 3 consecutive vertices of degree 2

To prove Theorem 2 we formulate and prove the following extension.

Theorem 4. *Every simple connected graph G has a spanning tree T, such that there is no path of length 2 in T all of whose vertices have degree 2 in T and degree at least 3 in G.*

This stronger version of Theorem 2 allows us to use induction on the size of G. The proof is split up into two parts: First, we reduce degree 1 and degree 2 vertices showing that a minimum counterexample to Theorem 4 has minimum degree at least 3. The second part is to find reducible structures in graphs of minimum degree at least 3. Thomassen and Toft [7] proved that every connected graph G with minimum degree at least 3 contains an induced cycle C such that $G-V(C)$ is connected. Let G be a minimal counterexample to Theorem 4. By the above G has minimum degree at least 3, so there exists an induced cycle C for which $G - V(C)$ is connected. It can be checked that such a cycle is a reducible configuration unless it contains vertices of degree at least 4 in G. This already shows that Theorem 4 is true for subcubic graphs and Theorem 2 for cubic graphs. To prove Theorem 4 in its full generality, we prove Lemma 2 below which allows us to find other reducible structures in the case where all non-seperating cycles contain vertices of degree at least 4. The so-called *W_a-configurations* and *$W_{a,b}$-configurations* are explained in Figure 2.



(a) A W_a-configuration

(b) A $W_{a,b}$-configuration

Figure 2: Two types of graph-configurations.

Lemma 2. *Let G be a connected graph of minimum degree at least 3. Let S be a set of vertices in G containing all vertices of degree greater than 3 and possibly some vertices of degree 3. Then at least one of the following three conditions is satisfied:*

(C) *There exists an induced cycle C containing no vertex of S such that $G - E(C)$ is connected.*

(P) *There exist an induced path P with endvertices in S such that $G - E(P)$ is connected.*

(W) *There exists a W_a-configuration or a $W_{a,b}$-configuration in G where the vertex v is contained in S.*

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