On kissing numbers in high dimensions

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Abstract

We prove a lower bound of $\Omega(d^{3/2} \cdot (2/\sqrt{3})^d)$ on the kissing number in dimension d. This improves the classical lower bound of Chabauty, Shannon, and Wyner by a linear factor in the dimension. We obtain a similar linear factor improvement to the best known lower bound on the maximal size of a spherical code of acute angle θ in high dimensions.

1 Kissing numbers

The kissing number in dimension d, $K(d)$, is the maximum number of non-overlapping unit spheres that can touch a single unit sphere in dimension d. The kissing number has been determined exactly in only a small number of dimensions: $1, 2, 3, 4, 8$, and 24 [8, 10, 11, 14].

For very large dimensions, much less is known. There is a classical lower bound, due in various forms to Chabauty [2], Shannon [15], and Wyner [19], that shows

$$K(d) \geq (1 + o(1))\sqrt{\frac{3\pi d}{8}}\left(\frac{2}{\sqrt{3}}\right)^d. \tag{1}$$

The argument is very elegant. The centers of the kissing spheres, when projected radially onto the unit sphere, must be at angular distance at least $\pi/3$ from each other. Now suppose we have an optimal kissing configuration in dimension d. Then it must be the case that the spherical caps of angular radius $\pi/3$ cover the unit sphere, otherwise we could add another kissing sphere. The bound in (1) is simply the reciprocal of the fraction of the surface of the unit sphere covered by a cap of angular radius $\pi/3$. We will call this the covering lower bound.

There is an equally simple upper bound of 2^d on an exponential scale, by a volume argument: the spherical caps of angular radius $\pi/6$ around the projections of the centers of the kissing spheres must be disjoint. This was improved by Rankin [12] to

$$K(d) \leq (1 + o(1))\sqrt{\frac{\pi}{8}} d^{3/2} \cdot 2^{d/2}.$$

Rankin's bound was further improved by the breakthrough work of Kabatyanskii and Levenshtein [7] who applied the method of Delsarte [5] to obtain

$$K(d) \leq 2^{.4041\ldots \cdot d}.$$

To the best of our knowledge there have been no further improvements to either the covering lower bound or the Kabatyanskii and Levenshtein upper bound, and the gap remains exponentially large in d.

Our main result is a linear factor improvement to the lower bound.

Theorem 1. *As $d \to \infty$, we have*

$$K(d) \geq (1 + o(1))\sqrt{\frac{3\pi}{8}} \log\frac{3}{2\sqrt{2}} \cdot d^{3/2}\left(\frac{2}{\sqrt{3}}\right)^d.$$

The constant $\sqrt{\frac{3\pi}{8}} \log \frac{3}{2\sqrt{2}}$ is approximately .0639.

It is instructive to compare the state of affairs of the kissing numbers in high dimensions to that of the maximum sphere packing density in $\mathbb{R}^d$. As with the kissing numbers there is a very simple covering argument that gives a lower bound of 2^{-d} (this is attributed to Minkowski [9] who proved a slightly better lower bound of $\zeta(d) \cdot 2^{1-d}$). There is an upper bound of $2^{-.599..d}$ due to Kabatyanskii and Levenshtein [7] based on their bound for spherical codes. No improvements on an exponential scale to either bound are known.

There are, however, several important works providing improvements on a smaller scale. Most significantly, Rogers [13] improved the asymptotic order of the lower bound by showing a lower bound of $\Omega(d \cdot 2^{-d})$. Rogers obtained this improvement by analyzing a random lattice packing of $\mathbb{R}^d$ by means of the Siegel mean-value theorem. Subsequent work of Davenport-Rogers [4], Ball [1], Vance [16], and Venkatesh [17] has improved the leading constant of Rogers' result. Venkatesh also obtained an additional factor of $\log \log d$ in a sparse sequence of dimensions. Finally, Cohn and Zhao [3] recently obtained a constant factor improvement to the Kabatyanskii and Levenshtein upper bound, by improving the geometric argument that links spherical codes to sphere packings.

Theorem 1 is an analogue of Rogers' result: both give a linear factor improvement to the simple covering lower bound. As far as we understand, none of the lattice-based methods used for the sphere packing lower bounds mentioned above can be adapted to kissing numbers. In fact, only very recently has it been shown by Vlăduţ that the lattice kissing number is exponential in d [18] (albeit with a smaller base of the exponent than $2/\sqrt{3}$). Achieving the bound in Theorem 1 or (1) for the lattice kissing number remains a challenging open problem.

2 Spherical codes

As alluded to above, kissing configurations are examples of *spherical codes*. Let S_{d-1} denote the unit sphere in dimension d. A spherical code of angle θ in dimension d is a set of unit vectors $x_1, \ldots, x_k \in S_{d-1}$ so that $\langle x_i, x_j \rangle \leq \cos\theta$ for all $i \neq j$; that is, the angle between each pair of distinct vectors is at least θ. The size of such a spherical code is k, the number of these unit vectors, or *codewords*. We denote by $A(d, \theta)$ the size of the largest spherical code of angle θ in dimension d. The kissing number can therefore be written as $K(d) = A(d, \pi/3)$. For $\theta \geq \pi/2$, Rankin [12] determined $A(d, \theta)$ exactly. In what follows we will therefore consider $\theta \in (0, \pi/2)$ fixed and use standard asymptotic notation, $O(\cdot), \Omega(\cdot), \Theta(\cdot), o(\cdot)$, all as $d \to \infty$.

For a measurable set $A \subseteq S_{d-1}$, let $s(A)$ denote the normalized surface area of A; that is $s(A) = \frac{\hat{s}(A)}{\hat{s}(S_{d-1})}$ where $\hat{s}(\cdot)$ is the usual surface area. For $x \in S_{d-1}$, let $C_\theta(x)$ be the spherical cap of angular radius θ around x; that is, $C_\theta(x) = \{y \in S_{d-1} : \langle x, y \rangle \geq \cos\theta\}$. Let $s_d(\theta) = s(C_\theta(x))$ be the normalized surface area of a spherical cap of angular radius θ; in other words,

$$s_d(\theta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(d/2)}{\Gamma((d-1)/2)} \int_0^\theta \sin^{d-2} x \, dx \, .$$

The covering argument of Chabauty [2], Shannon [15], and Wyner [19] gives a general lower bound for the size of spherical codes:

$$A(d, \theta) \geq \frac{1}{s_d(\theta)} = (1 + o(1)) \sqrt{2\pi d} \cdot \frac{\cos\theta}{\sin^{d-1}\theta} \, . \tag{2}$$

The kissing number bound (1) is an application of (2) with $\theta = \pi/3$.

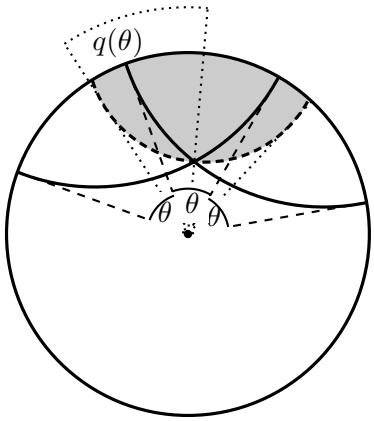


Figure 1: The angle $q(\theta)$.

The best upper bound on $A(d,\theta)$ is due to Kabatyanskii and Levenshtein [7]:

$$A(d,\theta) \leq e^{\phi(\theta)d(1+o(1))},$$

for a certain $\phi(\theta) > -\log\sin\theta$. Again to the best of our knowledge there have been no further improvements to these bounds, and so for every $\theta \in (0,\pi/2)$ the gap between the upper and lower bounds is exponential in d.

As with the kissing number, we improve the lower bound on the maximal size of spherical codes by a linear factor in the dimension. To state our result, we define $q(\theta)$ to be the angular radius of the smallest spherical cap that contains the intersection of two spherical caps of angular radius θ whose centers are at angle θ (see Figure 1); that is,

$$q(\theta) = \arcsin\left(\frac{\sqrt{(\cos\theta-1)^2(1+2\cos\theta)}}{\sin\theta}\right). \tag{3}$$

For instance, $q(\pi/3) = \arcsin(\sqrt{2/3})$. Crucially for our proof, $q(\theta) < \theta$ for all $\theta \in (0,\pi/2)$.

Theorem 2. *Let $\theta \in (0,\pi/2)$ be fixed. Then*

$$A(d,\theta) \geq (1+o(1))\frac{c_\theta \cdot d}{s_d(\theta)}$$

as $d \to \infty$, where $c_\theta = \log\left(\frac{\sin\theta}{\sin q(\theta)}\right) = \log\frac{\sin^2\theta}{\sqrt{(1-\cos\theta)^2(1+2\cos\theta)}}$.

Theorem 1 is obtained from Theorem 2 by setting $\theta = \pi/3$. A full version of the paper can be found here [6].

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