# On $S$-packing edge-colorings of cubic graphs

**Nicolas Gastineau** — **Le2i, Université de Bourgogne Franche-Comté, Dijon, France**
**Olivier Togni** — **Le2i, Université de Bourgogne Franche-Comté, Dijon, France**

**Abstract**

Given a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, an *$S$-packing edge-coloring* of a graph $G$ is a partition of the edge set of $G$ into $k$ subsets $\{X_1, X_2, \ldots, X_k\}$ such that for each $1 \leq i \leq k$, the distance between two distinct edges $e, e' \in X_i$ is at least $s_i + 1$. This work studies $S$-packing edge-colorings of cubic graphs. Among other results, we prove that cubic graphs having a 2-factor are $(1, 1, 1, 3, 3)$-packing edge-colorable.

## 1 Introduction

All the graphs considered in this extended abstract are simple and connected, unless stated otherwise. A proper *edge-coloring* of a graph $G$ is a mapping which associates a color (an integer) to each edge such that adjacent edges get distinct colors. In such a coloring, each color class is a matching (also called stable set of edges or 1-packing). According to Vizing's famous theorem, every cubic graph needs either 3 or 4 colors for a proper edge-coloring. The bridgeless cubic graphs (often with other restrictions) which are not edge-colorable with three colors are called *snarks* [3, 9].

As an extension, a $d$-strong edge-coloring of $G$ is a proper coloring such that edges at distance at most $d$ have distinct colors, i.e., a partition of $E(G)$ into sets of edges at pairwise distance at least $d + 1$, also called $d$-packings. A 2-strong edge-coloring is simply called a strong edge-coloring and a 2-packing of edges is an induced matching. Strong edge-colorings of cubic graphs retain a lot of attention since decades [1, 4, 8].

The aim of this work is to study a mixing of these two types of edge-colorings, i.e., colorings of (sub)cubic graphs in which some color classes are 1-packings while other are $d$-packings, $d \geq 2$. More formally, given a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, an *$S$-packing edge-coloring* of a graph $G$ is a partition of the edge set of $G$ into $k$ subsets $\{X_1, X_2, \ldots, X_k\}$ such that each $X_i$ is an $s_i$-packing, $1 \leq i \leq k$.

For an edge-coloring, a color for which the color class is an $r$-packing is said to be a color of *radius* $r$. In order to avoid long subsequences of the same integer in sequences of colors, we sometimes use the exponent to denote repetitions of an integer, e.g., $(1^2, 2^5) = (1, 1, 2, 2, 2, 2, 2)$. Also, to simplify, an $S$-packing edge-coloring will be simply called an *$S$-coloring* in the remainder of the paper. A $(1, 1, 1, 2)$-coloring and a $(1, 1, 2, 2, 2)$-coloring of the Petersen graph are illustrated in Figure 1 (one can check that the Petersen graph is not $(1, 1, 2, 2)$-colorable).

Let $G$ be a graph and $A \subseteq E(G)$. By $G^k[A]$, we denote the graph with vertex set $A$ and edge set $\{ee' \in E(G) |\ e \in A,\ e' \in A,\ d_G(e, e') \leq k\}$, where $d_G(e, e')$ is the usual distance between the two edges $e$ and $e'$ in $G$. We recall that a 2-*factor* of $G$ is a spanning subgraph of $G$ that consists in a disjoint union of cycles. For a cubic graph $G$ having a 2-factor, the *oddness* of $G$ is the minimum number of odd cycle among all 2-factors of $G$. According to Petersen's theorem, every bridgeless cubic graph has a 2-factor.

**Definition 1.1.** *For a graph $G$ with a 2-factor $\mathscr{F}$ and a set $A \subseteq E(\mathscr{F})$, $A$ is of type I if it contains exactly one edge per odd cycle of $\mathscr{F}$ and no edge of any even cycle of $\mathscr{F}$;*

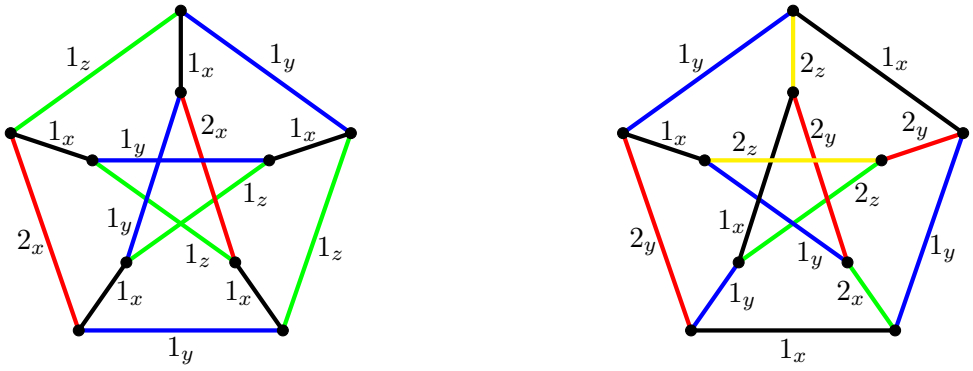



Figure 1: A $(1,1,1,2)$-coloring (on the left) and a $(1,1,2,2,2)$-coloring (on the right) of the Petersen graph.

Table 1: The minimum integer $n = \ell + m$ for which all cubic graphs (and all 3-edge-colorable cubic graphs) having a 2-factor are $(1^\ell, k^m)$-colorable (the bold numbers represent the exact values of $n$ and a pair of two integers $a$-$b$ represents a lower bound and an upper bound on $n$).

| Class | cubic graphs | | | 3-edge-colorable cubic graphs | | |
|---|---|---|---|---|---|---|
| $k \backslash \ell$ | 1 | 2 | 3 | 1 | 2 | 3 |
| 2 | 8-10 | 6-7 | **4** [6] | 6-9 | 5-6 | **3** |
| 3 | 15-21 | 9-13 | **5** | 15-19 | 9-11 | **3** |
| 4 | 31-48 | 17-28 | 5-8 | 31-43 | 17-23 | **3** |

As any subcubic graph $H$ is the subgraph of a cubic graph $G$ and as $d_H(e,e') \geq d_G(e,e')$ for any two edges $e, e' \in E(H)$, then any $S$-coloring of $G$ is also an $S$-coloring of $H$. Therefore, the results of this extended abstract that are not concerned with oddness can be easily extended to subcubic graphs.

Table 1 summarizes the main results proven in the extended version of this work [7]. In this table, every non-referenced result of the five first column is new.

## 2 Sets of type I

The following result from Fouquet and Vanherpe is a structural property about non 3-edge-colorable cubic graphs [6]. It will be used several times in the proofs of Section 3.

**Proposition 2.1** ([6]). *Let $G$ be a cubic graph having a 2-factor. Let $\mathscr{F}$ be a 2-factor of $G$ containing a minimum number of odd cycles and let $\mathscr{F}'$ be the set of odd cycles from $\mathscr{F}$. Then, no three edges in different cycles of $\mathscr{F}'$ induce in $G$ a subgraph containing a path of length 5.*

Note that the previous proposition was not explicitly presented in the paper of Fouquet and Vanherpe but can be easily obtained by combining Properties 3 and 6 of [6, Theorem 4]. Note also that the proof of Theorem 4 is written in another paper [5] from the same authors.

**Lemma 2.2** ([6]). *Let $G$ be a cubic graph having a 2-factor. Let $\mathscr{F}$ be a 2-factor of $G$ containing a minimum number of odd cycles and let $\mathscr{F}'$ be the set of odd cycles from $\mathscr{F}$. There exists a set $A$ of type I in $\mathscr{F}'$ such that $G^2[A]$ is an empty graph, i.e., $\chi(G^2[A]) \leq 1$.*

As the previous proposition, Lemma 2.2 has not been explicitly presented in the paper of Fouquet and Vanherpe but is an intermediate step to prove a Theorem [6, Theorem 8].

The following remark will allow to reduce the problem of finding an $S$-coloring to the one of finding a set $A$ of type I such that $G^k[A]$ has small chromatic number (where $2 \leq k \leq 4$ and $k$ appears in $S$).

**Remark 2.3.** *Let $G$ be a cubic graph having a 2-factor $\mathscr{F}$ and let $\ell$ be a positive integer. Let $A \subseteq E(\mathscr{F})$ be of type I. If $\chi(G^k[A]) \leq \ell$, then $G$ is $(1,1,1,k^\ell)$-colorable.*

## 3 $(1,1,1,k,\ldots,k)$-coloring

For the case $k = 2$, Payan [10] has shown that one color of radius two is sufficient. Another proof of this result has been given by Fouquet and Vanherpe [6] (Lemma 2.2 is an intermediate step of the proof of this result).

**Theorem 3.1** ([6, 10]). *Every cubic graph is $(1,1,1,2)$-colorable.*

Notice that this result is tight since the Petersen graph is not $(1,1,1,3)$-colorable (as it is not 3-edge-colorable and has diameter 2).

**Theorem 3.2.** *Every cubic graph having a 2-factor is $(1,1,1,3,3)$-colorable.*

*Proof.* Let $G$ be a cubic graph. Let $\mathscr{F}$ be a 2-factor of $G$ having a minimum number of odd cycles. Let $\mathscr{F}'$ be the set of odd cycles from $\mathscr{F}$. By Remark 2.3, if there exists a set $A$ of type I in $\mathscr{F}'$ such that $\Delta(G^3[A]) \leq 1$, then $G$ is $(1,1,1,3,3)$-colorable.

We give labels to the vertices of $\mathscr{F}'$ as follows. If a vertex belonging to an odd cycle $C$ from $\mathscr{F}'$ has a neighbor in a different cycle of $\mathscr{F}'$, we label it by $+$, otherwise we label it by $-$. By Proposition 2.1, the two end vertices of any edge of any cycle $C$ of $\mathscr{F}'$ have neighbors in only at most one cycle of $\mathscr{F}'$ other than $C$. Thus, if consecutive vertices are labeled by $+$ in an odd cycle $C$ from $\mathscr{F}'$, then there exists an unique cycle $C'$ of $\mathscr{F}'$ such that all these vertices only have neighbors in $C \cup C'$.

**Observation 1.** For each edge $e \in E(G - \mathscr{F})$ having an extremity $u$ in a odd cycle $C$ from $\mathscr{F}'$ the following is true:

i) if $u$ is labeled by $+$, then all edges in $\mathscr{F}'$ at distance at most 2 from $e$ are included in $C \cup C'$, for $C'$ an odd cycle of $\mathscr{F}'$;

ii) if $u$ is labeled by $-$, then all edges in $\mathscr{F}'$ at distance at most 2 from $e$ are included in $C$.

The previous observation can be easily obtained using the fact that $e$ is only adjacent with edges of $\mathscr{F}$, these edges being themselves adjacent with edges either in the same cycle than they or in $G - \mathscr{F}$.

We will construct $A$, starting from an empty set, as follows. Since each cycle $C$ of $\mathscr{F}'$ has an odd number of vertices there exist two consecutive vertices both labeled either by $+$ or by $-$ in

every cycle of $\mathscr{F}'$. Let $u_1$ and $u_2$ be these two adjacent vertices (both labeled either by $+$ or $-$) and suppose that $u_0$, $u_1$, $u_2$, $u_3$ and $u_4$ are consecutive vertices of the cycle $C$ (if $C$ contains three vertices then $u_3 = u_0$ and $u_4 = u_1$). For each cycle $C$ of $\mathscr{F}'$, we add to $A$ an edge of $C$ depending on the label of $u_1$ and $u_2$.

If $u_1$ and $u_2$ are both labeled by $+$ in $C$, then we add the edge $u_1u_2$ into $A$. Note that $u_0, u_1, u_2, u_3$ are labeled either by $+,+,+,+$, by $-,+,+,+$, by $+,+,+,-$ or by $-,+,+,-$ (the labels are given following the index of $u$). Consequently, by Observation 1, there exists a cycle $C'$ from $\mathscr{F}'$ such that all edges from $\mathscr{F}'$ at distance at most 3 from $u_1u_2$ are in $C \cup C'$.

If $u_1$ and $u_2$ are both labeled by $-$, then we add the edge $u_2u_3$ to $A$. Note that $u_1, u_2, u_3, u_4$ are labeled either by $-,-,+,+$, by $-,-,-,+$, by $-,-,+,-$ or by $-,-,-,-$. Also in this case, by Observation 1, there exists a cycle $C'$ from $\mathscr{F}'$ such that all edges from $\mathscr{F}'$ at distance at most 3 from $u_1u_2$ are in $C \cup C'$.

Since there is one edge of $A$ per cycle of $\mathscr{F}'$ we obtain, by construction, that $\Delta(G^3[A]) \leq 1$ and thus $\chi(G^3[A]) \leq 2$. Finally, by Remark 2.3, $G$ is $(1,1,1,3,3)$-colorable. □

We finish this extended abstract by giving one result (without the proof) that come from the extended version of this work [7]:

**Proposition 3.3.** *Every cubic graph of oddness* 2 *having a* 2*-factor containing two odd cycles* $C_1$ *and* $C_2$ *is* $(1,1,1,3)$*-colorable in the case* $C_1$ *or* $C_2$ *is a cycle of length at least* 13*. A consequence is that the Petersen graph is the only generalized Petersen graph which is not* $(1,1,1,3)$*-colorable and the Tietze graph is the only flower snark which is not* $(1,1,1,3)$*-colorable.*

## References


[1] L. D. Andersen, The strong chromatic index of a cubic graph is at most 10, *Discrete Mathematics*, 108(1): 231–252, 1992.

[2] J. Balogh, A. Kostochka, X. Liu, Packing chromatic number of cubic graphs, *Discrete Mathematics*, 341(2): 474–483, 2018.

[3] G. Brinkmann, J. Goedgebeur, J. Hägglund, K. Markström, Generation and properties of snarks, *Journal of Combinatorial Theory*, Series B, 103(4): 468–488, 2013.

[4] J. L. Fouquet, J. L. Jolivet, Strong edge-colorings of graphs and applications to multi-k-gons, *Ars Combinatoria*, 16A: 141–150, 1983.

[5] J. L. Fouquet, J. L. Jolivet, Tools for parsimonious edge-coloring of graphs with maximum degree three, *on arXiv*: 1201.5988v1, 2012.

[6] J. L. Fouquet, J. M. Vanherpe, On parsimonious edge-colouring of graphs with maximum degree three, *Graphs and Combinatorics*, 29(3): 475-487, 2013.

[7] N. Gastineau, O. Togni, On $S$-packing edge-colorings of cubic graphs *arXiv:1711.10906*, 2017.

[8] H. Hocquard, M. Montassier, A. Raspaud, P. Valicov, On strong edge-colouring of subcubic graphs, *Discrete Appl. Math.*, 161(16): 2467–2479, 2013.

[9] R. Lukoťka, E. Máčajová, J. Mazák, M. Škoviera, Small snarks with large oddness, *The Electronic Journal of Combinatorics*, 22(1): #P1.51, 2015.

[10] C. Payan, Sur quelques problèmes de couvertures et de couplages en combinatoire. Thèse d'état (in french), 1977.