Even Cycles in Dense Graphs

Neal Bushaw — **Virginia Commonwealth University**
Andrzej Czygrinow — **Arizona State University**
Jangwon Yie — **Arizona State University**

Abstract

Understanding the cycle structure of graphs has long been of great interest to graph theorists. For instance, it is elementary to show that every graph contains a cycle of length at least its minimum degree (this has been proven in the first week of every graph theory course I've ever attended). Adding a connectivity condition gives much more: Dirac's Theorem says that every 2-connected graph of minimum degree d contains a cycle of length min2d,—G—. When we add a 2-connectivity condition, our graphs not only contain long cycles, but cycles of many different lengths. And so, we must explore the cycle spectrum of 2-connected graphs: the set of cycle lengths appearing in the graph. In this talk, we'll tell a little bit of the history of results on the cycle spectrum, as well as present some new results on the even cycle spectrum of large 2-connected graphs (in particular, proving in a strong form the dense case of a conjecture of Faudree, Gould, Jacobson, and Magnant).

Determining the structure of cycles in graphs is a problem of fundamental interest in graph theory; this thread traces through numerous subareas in structural and extremal graph theory. For a graph G we use $c(G)$ to denote the circumference of G, $oc(G)$ ($ec(G)$) to denote the length of the longest odd (even) cycle in G. If G is a graph of minimum degree d, then $c(G) \geq d$ which is the best possible. However, additional assumptions on the connectivity of G usually lead to better bounds for $c(G)$ or $ec(G)$ and $oc(G)$. For example, Dirac's theorem states that if G is a 2-connected graph on n vertices, then $c(G) \geq \min\{n, 2\delta(G)\}$.

Dirac's Theorem gave birth to a large body of research centered around determining the length of the longest cycle in a graph satisfying certain conditions. Indeed, one could even search for graphs which contain cycles of all possible lengths. Such graphs are called pancyclic, and they, too, are well studied. Bondy observed that in many cases a minimum degree which implies the existence of a spanning cycle also implies that the graph is pancyclic. It's natural to ask if analogous statements are true for graphs with smaller minimum degree. In [2], Gould et. al. proved the following result.

Theorem 1 (Gould, Haxell and Scott)**.** *For every $\alpha > 0$ there is K such that if G is graph on $n > 45K\alpha^{-4}$ vertices with $\delta(G) \geq \alpha n$, then G contains cycles of every even length from $[4, ec(G) - K]$ and every odd length from $[K, oc(G) - K]$.*

Nikiforov and Shelp ([3]) proved that if G is a graph on n vertices with $\delta(G) \geq \alpha n$, then G contains cycles of every even lengths from $[4, \delta(G) + 1]$ as well as cycles of odd lengths from $[2k - 1, \delta(G) + 1]$, where $k = \lceil 1/\alpha \rceil$, unless G is one of several explicit counterexamples.

One of the motivations for our work is the following conjecture of Faudree, Gould, Jacobson and Magnant [1] on cycle spectra. Let $S_e = \{|C| : C \text{ is an even cycle contained in } G\}$ and $S_o = \{|C| : C \text{ is an odd cycle contained in } G\}$.

Theorem 2 (Conjecture)**.** *If G is 2-connected graph, then $\delta(G) = d \geq 3$ implies that $|S_e| \geq d - 1$, and, if, in addition G is not bipartite, then it implies that $|S_o| \geq d$.*

In [1], this conjecture was confirmed for $d = 3$.

We prove that for almost all values of $n_1, \ldots, n_l$ such that $\sum n_i = \delta(G)$, G indeed contains C where C is the union of disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$. There are however two obstructions, of which one is well-known, when G is a graph on n vertices with minimum degree which satisfies $\alpha n < \delta(G) < (n-1)/2$. These are explicit examples, although we avoid describing them in this note. The main result of the paper is the following.

Theorem 3. *For every $0 < \alpha < \frac{1}{2}$, there is a natural number $N = N(\alpha)$ such that the following holds. For any $n_1, ..., n_l \in 2Z^+$ such that $\sum_{i=1}^{l} n_i = \delta(G)$, every 2-connected graph G of order $n \geq N$ and $\alpha n \leq \delta(G) < n/2 - 1$ contains C where C is a disjoint union of $C_{2n_1}, \ldots, C_{2n_l}$ or G is one of the two obstructions mentioned above.*

References

[1] R. Faudree, R. Gould, M. Jacobson, C. Magnant. Minimum degree and even cycle lengths. Manuscript.

[2] R. J. Gould, P. E. Haxell, A. D. Scott. A note on cycle lengths in graphs. Graphs and Combinatorics, 18, (2002), 491-498.

[3] V. Nikiforov, R. H. Schelp. Cycle lengths in graphs with large minimum degree. Journal of Graph Theory, 52, (2006), 157-170.