Semistrong matching and semistrong edge-coloring

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Abstract

A proper edge-coloring of a graph in which every color class is an induced matching is called a *strong edge-coloring*. Here, we consider a related edge-coloring with relaxed conditions. A matching M of a graph G is *semistrong* if every edge of M has an end-vertex of degree one in the induced subgraph $G[M]$. A *semistrong edge-coloring* of a graph G is a proper edge-coloring in which every color class induces a semistrong matching. This notion was initiated by Gyárfás and Hubenko in 2005, who determined the equality between the sizes of maximum induced matchings and maximum semistrong matchings in Kneser and subset graphs.

We present a survey of the two types of edge-colorings, give the upper bound for the semistrong chromatic index of general graphs, and tight bounds for trees and graphs with maximum degree 3.

Besides the notion of a matching of a graph G, which is a set of pairwise nonadjacent edges of G, several other types of more restricted matchings are known. A matching M of a graph G is a *strong* or an *induced matching* if no end-vertices of two edges of M are joined by an edge of G. A proper edge-coloring of a graph G is a partition of the edges of G into matchings, hence a corresponding edge-coloring for every type of matching can be studied. A proper edge-coloring of a graph G in which the edges of every color class form an induced matching is called a *strong edge-coloring*. In such a coloring, any two edges at distance at most 2 are assigned distinct colors. The minimum number of colors for which G admits a strong edge-coloring is the *strong chromatic index* of G, denoted by $\chi'_s(G)$.

While the chromatic index $\chi'(G)$ attains one of just two values, $\Delta(G)$ or $\Delta(G)+1$, the interval of possible values for $\chi'_s(G)$ is much wider. The upper bound was conjectured by Erdős and Nešetřil in 1985, and they also gave a construction of graphs for which their strong chromatic index attains this bound.

Conjecture 1 (Erdős, Nešetřil, 1985)**.** *For every graph G with maximum degree $\Delta(G)$,*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2, & \Delta(G) \text{ is even,} \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1), & \Delta(G) \text{ is odd.} \end{cases}$$

The first general upper bound on strong chromatic index was established by Molloy and Reed [7], who used the probabilistic argument to prove that $1.998\Delta(G)^2$ colors suffice for any graph with maximum degree large enough. The bound was further improved by Bruhn and Joos [3], and finally to $1.835\Delta(G)^2$ by Bonamy et al. [2].

While Conjecture 1 trivially holds for isolated edges and paths, it remains open for most of more complex graph classes. It has been confirmed for subcubic graphs by Andersen [1] and independently by Horák et al. [6]. A subcubic graph achieving the bound of 10 colors is e.g. $K_{3,3}$ with one subdivided edge.

Theorem 1 (Andersen, 1992; Horák et al., 1993)**.** *For every subcubic graph G it holds $\chi'_s(G) \leq 10$.*

Since the beginning, study of strong edge-coloring has become a topic of interest for many authors, and has developed in several directions. Perhaps the most intense are considerations of bipartite graphs, sparse graphs, and above mentioned lowering the bound for general graphs. For bipartite graphs, Faudree et al. [4] conjectured that $\Delta(G)^2$ colors suffice (again, the bound would be tight by complete bipartite graphs), and Steger and Yu [8] verified it for subcubic bipartite graphs. For planar graphs, it was proved in [4] that at most $4\Delta(G)+4$ colors always suffice, while there exist graphs with strong chromatic index equal to $4\Delta(G)-4$.

Richness of the field motivated studying of matchings similar to strong matchings and the corresponding chromatic indices. In this paper, we continue the work initiated by Gyárfás and Hubenko [5] on semistrong matching and semistrong edge-coloring. Their motivation came from the problem of finding the smallest n for which a graph H is an induced subgraph of n-dimensional cube Q_n, where H is a union of paths and even cycles. They define a matching M of a graph G to be *semistrong* if every edge of M has an end-vertex of degree one in the induced subgraph $G[M]$. Clearly, every strong matching is also semistrong, hence for every graph G it holds that

$$\nu_s(G) \leq \nu_{ss}(G) \leq \nu(G),$$

where by $\nu(G)$, $\nu_s(G)$ and $\nu_{ss}(G)$ we denote the maximum size of a matching, strong matching and semistrong matching of G, respectively.

A *semistrong edge-coloring* is a proper edge-coloring in which every color class is a semistrong matching, and has hence a weaker condition regarding the distance between edges of the same color as the strong edge-coloring. The least integer k such that G admits a semistrong edge-coloring with at most k colors is the *semistrong chromatic index* of G and denoted by $\chi'_{ss}(G)$.

Although establishing the series of inequalities

$$\chi'(G) \leq \chi'_{ss}(G) \leq \chi'_s(G),$$

for every graph G, is straightforward, no better general upper bound than the bound for strong edge-coloring has been known for semistrong edge-coloring of a graph. Here, we present currently the best upper bound for general graphs.

Theorem 2. *For every graph G it holds*

$$\chi'_{ss}(G) \leq \Delta(G)^2\,.$$

Let us note, there are classes of graphs, for which the two bounds coincide, e.g., the complete and the complete bipartite graphs. The equality is simply derived from the fact that in semistrong edge-coloring the edges of every 4-cycle must be colored with four distinct colors.

More surprising is perhaps that apart from the two complete graph families, there are other families of graphs with the same maximum size of strong and semistrong matchings and moreover even the same value of strong and semistrong chromatic indices. In [5], the authors found two such families, namely Kneser and subset graphs.

Given two positive integers n and k, a *Kneser graph* $K(n,k)$ is a graph whose vertex set consists of k-subsets of an n element set (for $n > 2k$ to be connected), and two vertices are connected if and only if the corresponding sets are disjoint. A *subset graph* $S_n(k,l)$, for $0 \leq k \leq l \leq n$, is a bipartite graph where vertex sets are the k- and l-subsets of an n element set, and two vertices (subsets) are connected if and only if one of them is contained in the other. It was proved in [4] that $\chi'_s(K(n,k)) = \binom{n}{2k}$, and in [9] that $\chi'_s(S_n(k,l)) = \binom{n}{l-k}$.

Theorem 3 (Gyárfás and Hubenko, 2005)**.**

(*a*) *For every Kneser graph* $K(n,k)$ *it holds* $\chi'_s(K(n,k)) = \chi'_{ss}(K(n,k)) = \binom{n}{2k}$.

(*b*) *For every subset graph* $S_n(k,l)$ *it holds* $\chi'_s(S_n(k,l)) = \chi'_{ss}(S_n(k,l)) = \binom{n}{l-k}$.

The question of equality of the parameters remains open for n-dimensional cubes. It was proved in [4] that $\nu_s(Q_n) = 2^{n-2}$ and $\chi'_s(Q_n) = 2n$. The conjecture was proposed in [5].

Conjecture 2 (Gyárfás and Hubenko, 2005)**.** *For every integer* $n \geq 2$ *it holds*

$$\nu_s(Q_n) = \nu_{ss}(Q_n) = 2^{n-2}.$$

The topic of our interest are also trees and graphs with maximum degree 3. The strong chromatic index of a tree T is equal to $2\Delta(T) - 1$ [4]. For the semistrong edge-coloring of trees, the bound is much lower.

Theorem 4. *For every tree* T *it holds*

$$\chi'_{ss}(T) \leq \Delta(T) + 1\,.$$

Moreover, if T *has diameter at most* 4*, then*

$$\chi'_{ss}(T) = \Delta(T)\,.$$

From previous results it follows that the bound on semistrong chromatic index of any graph G with maximum degree 3 lies between 4 and 10. We prove the following.

Theorem 5. *For every connected subcubic graph* G *distinct from* $K_{3,3}$ *it holds*

$$\chi'_{ss}(G) \leq 8\,.$$

If G *is isomorphic to* $K_{3,3}$*, then* $\chi'_{ss}(G) = 9$.

Note that the bound is tight and it is attained by the graph of 5-prism, as the size of a maximum semistrong matching in the 5-prism is 2 and there are 15 edges. We further believe that the established bound can be improved further as follows.

Conjecture 3. *For every connected subcubic graph* G *distinct from* $K_{3,3}$ *and the* 5*-prism it holds*

$$\chi'_{ss}(G) \leq 7\,.$$

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References

[1] L. D. Andersen. The strong chromatic index of a cubic graph is at most 10. Discrete Math. 108 (1992), 231–252.

[2] M. Bonamy, T. Perrett, L. Postle. Colouring graphs with sparse neighbourhoods: Bounds and applications. Manuscript.

[3] H. Bruhn and F. Joos. A stronger bound for the strong chromatic index. Electron. Notes Discrete Math. 49 (2015), 277–284.

[4] R. J. Faudree, A. Gyárfás, R. H. Schelp, Zs. Tuza. The strong chromatic index of graphs. Ars Combin. 29B (1990), 205–211.

[5] A. Gyárfás and A. Hubenko. Semistrong edge coloring of graphs. J. Graph Theory 49 (2005), 39–47.

[6] P. Horák, H. Qing, W. T. Trotter. Induced matchings in cubic graphs. J. Graph Theory 17(2) (1993), 151–160.

[7] M. Molloy and B. Reed. A bound on the strong chromatic index of a graph. J. Comb. Theory Ser. B 69 (1997), 103–109.

[8] A. Steger and M.-L. Yu. On induced matchings. Discrete Math. 120 (1993), 291–295.

[9] J. J. Quinn and A. T. Benjamin. Strong chromatic index of subset graphs. J. Graph Theory 24(3) (1997), 267–273.