Existence of Simple (q, k, λ) BIBDs with $q \equiv 1 \pmod{k(k-1)}$

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Abstract

We study the existence of simple (q, k, λ) BIBDs when the number of elements q is a prime power. We show that in many situations the necessary conditions $\lambda(q-1) \equiv 0 \mod (k-1)$, $\lambda q(q-1) \equiv 0 \mod k(k-1)$, and $\lambda \leq \binom{q-2}{k-2}$ are also sufficient for the existence of a simple (q, k, λ) BIBD. Here we focus on the cases when $q \equiv 1 \pmod{k(k-1)}$.

Let V be a finite set of symbols, and suppose $\mathcal{B}$ is a collection of subsets of V. Then $(V, \mathcal{B})$ is called a (v, k, λ) *balanced incomplete block design* (*BIBD*) if there are parameters v, k, and λ with $v > k \geq 2$ such that the following properties are satisfied: $|V| = v$; every *block* in $\mathcal{B}$ has exactly k symbols; every pair of distinct symbols appears in exactly λ blocks. Suppose $(V, \mathcal{B})$ is a (v, k, λ) BIBD, it holds that every symbol appears in exactly r blocks, where $r = \lambda(v-1)/(k-1)$, and $|\mathcal{B}| = b = vr/k$. So sometimes a BIBD is described as a (v, b, r, k, λ) design. A design without repeated blocks is called *simple*.

We obtain some existence theorems of simple BIBDs when the number of elements is a prime power. More specifically, we have the following results.

Theorem 1. *Let q be a prime power with $q \equiv 1 \pmod{k(k-1)}$. Then there is a simple (q, k, λ) BIBD for $\lambda \leq \binom{q-2}{k-2}$ in the following cases:*

1. *$k \in \{3, 4, 5, 6\}$, except when $(q, k) = (61, 6)$.*
2. *$k = 7$, except when $q = 43$ and possibly when either (a) $q \in \{127, 211, 31^6\}$ or (b) $q \in [261239791, 1.236597 \times 10^{13}]$ and $(-3)^{(q-1)/14} = 1$ in $GF(q)$.*
3. *$k \in \{8, 9\}$ and $q < 10000$, except possibly for $k = 8$, $q = 113, 169, 281, 337$, and $k = 9$, $q = 289, 361$.*

Theorem 2. *Let q be a prime power such that $q^2 + q + 1$ is a prime. Then there is a simple $(q^2 + q + 1, q + 1, \lambda)$ BIBD for $\lambda \leq \binom{q^2+q-1}{q-1}$.*

We summarize some of the valid pairs $(q^2 + q + 1, q + 1)$ as follows: $(7, 3)$, $(13, 4)$, $(31, 6)$, $(73, 9)$, $(307, 18)$, $(757, 28)$, $(1723, 42)$, $(3541, 60)$, $(5113, 72)$, $(8011, 90)$, $(10303, 102)$, $(17293, 132)$, $(28057, 168)$, $(30103, 174)$, $(86143, 294)$.

References

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