Tree-like distance colouring for planar graphs of sufficient girth

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Abstract

Given a multigraph G and a positive integer t, the distance-t chromatic index of G is the least number of colours needed for a colouring of the edges so that every pair of distinct edges connected by a path of fewer than t edges must receive different colours. Let $\pi'_t(d)$ and $\tau'_t(d)$ be the largest values of this parameter over the class of planar multigraphs and of (simple) trees, respectively, of maximum degree d. We have that $\pi'_t(d)$ is at most and at least a non-trivial constant multiple larger than $\tau'_t(d)$. We prove for odd t the existence of a quantity g depending only on t such that the distance-t chromatic index of any planar multigraph of maximum degree d and girth at least g is at most $\tau'_t(d)$ if d is sufficiently large. Such a quantity does not exist for even t. We also show a related, similar phenomenon for distance vertex-colouring.

Two classic theorems set a basis for our work.

The first is a result of Vizing from 1965 [19]: every (simple) planar graph of maximum degree d has chromatic index at most d, provided $d \geq 8$. (This statement was extended to $d = 7$ by Sanders and Zhao [17] but remains open for $d = 6$.) On the other hand, the Shannon multigraphs, i.e. triangles with edges of balanced multiplicity, are planar and have chromatic index a non-trivial factor (of 3/2) greater than d. Since the tree consisting of a single vertex with d neighbours has chromatic index d, a rough way to view this is as follows: with respect to edge-colouring, planar multigraphs resemble trees if cycles of length 2 are forbidden (for large enough maximum degree).

The second is a result of Grötzsch from 1959 [8]: every triangle-free planar graph has chromatic number at most 3. This is sharp due to the odd cycles. And so, even though there is a reduction by one in the number of colours required when the girth (the smallest cycle length) is at least 4, there is no fixed set of cycle lengths one could forbid to achieve a bound of 2, the optimal chromatic number over all trees.

We are curious how the above narrative extends when like-coloured elements must be at some minimum distance, and contemplate the following.

> *For distance colouring of planar (multi)graphs, when does some finite girth constraint ensure the problem resembles that of trees?*

As noted, a girth constraint of 3 suffices (and is optimal) for ordinary edge-colouring, while for ordinary vertex-colouring no such constraint is possible.

More formally, we study the following parameters. Fix a positive integer t. Let $G = (V, E)$ be a multigraph. The *distance-t chromatic index* (*number*) $\chi'_t(G)$ ($\chi_t(G)$) of G is the least number of colours needed for a colouring of the edges (vertices) so that every pair of distinct edges (vertices) connected by a path of fewer than t ($t + 1$) edges must receive different colours. For $t = 1$ these correspond to the usual chromatic index $\chi'(G)$ and chromatic number $\chi(G)$ of G. Moreover, if H^t represents the graph whose adjacency matrix is the t-th power of the adjacency matrix of H, then $\chi_t(G) = \chi(G^t)$ and $\chi'_t(G) = \chi(L(G)^t)$, where $L(G)$ denotes the line graph of G.

These natural strengthened colouring parameters have been studied for almost half a century [11]. They have attracted much attention particularly in the cases of χ'_2, also called the *strong*

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chromatic index, cf. e.g. [15], and χ_2, cf. e.g. [20, 9]. That is not only because of their obvious mathematical appeal, but also because of their applicability in various other domains, such as the approximation of sparse Hessian matrices [14], link and broadcast scheduling in wireless networks [13, 16], algorithmic hardness of approximation [12]. Our results offer basic insights into fundamental characteristics of these important graph parameters.

The low waterline for our study is when G is of highest possible girth, i.e. it is a tree. Although the structure of G^t or $L(G)^t$ can be difficult to comprehend for general G, it is easy to characterise the extremal behaviour of $\chi'_t(G)$ or $\chi_t(G)$ if G has no cycles. For $t \geq 1$ and $d \geq 3$, define

$$\tau'_t(d) := \begin{cases} \frac{1}{d-2}(2(d-1)^{t/2+1} - d) & 2 \mid t \\ \frac{1}{d-2}(d(d-1)^{(t+1)/2} - d) & 2 \nmid t \end{cases} \quad \text{and}$$

$$\tau_t(d) := \begin{cases} \frac{1}{d-2}(d(d-1)^{t/2} - 2) & 2 \mid t \\ \frac{1}{d-2}(2(d-1)^{(t+1)/2} - 2) & 2 \nmid t \end{cases}.$$

Proposition 1. *Let $t \geq 1$ and $d \geq 3$. There is a tree G of maximum degree d with $\tau'_t(d)$ edges ($\tau_t(d)$ vertices) such that $L(G)^t$ (G^t) is a clique. If G is a tree of maximum degree d, then $\chi'_t(G) \leq \tau'_t(d)$ and $\chi_t(G) \leq \tau_t(d)$.*

The examples in Proposition 1 are merely subgraphs of the infinite d-regular tree formed by including all vertices or edges within a suitable fixed distance of some root vertex or edge, depending on the parity of t.

The high waterline for us is when G is a planar (multi)graph of maximum degree d. In this case $\chi'_t(G)$ or $\chi_t(G)$ is still at most a constant factor greater than $\tau'_t(d)$ or $\tau_t(d)$, but it can well be a non-trivial factor greater.

Proposition 2. *Let $t \geq 1$. Let $\pi'_t(d)$ ($\pi_t(d)$) be the largest value of $\chi'_t(G)$ ($\chi_t(G)$) over the class of planar multigraphs of maximum degree d. Then $\limsup_{d\to\infty} \pi'_t(d)/\tau'_t(d) \in [3/2, \infty)$ and $\limsup_{d\to\infty} \pi_t(d)/\tau_t(d) \in [3/2, \infty)$.*

Bounds in Proposition 2 for distance vertex-colouring were established generally for all t in [1] (upper bounds) and [7] (lower bounds). It is difficult to determine the precise values in Proposition 2, especially for distance vertex-colouring: $\limsup_{d\to\infty} \pi_1(d)/\tau_1(d) = 2$ is the Four Colour Theorem [2, 3], while $\limsup_{d\to\infty} \pi_2(d)/\tau_2(d) = 3/2$ is the asymptotic confirmation of Wegner's Conjecture [9]. Also $\limsup_{d\to\infty} \pi'_1(d)/\tau'_1(d) = 3/2$ is Shannon's Theorem [18]. No precise value is known for any $t \geq 3$.

Our main result resolves the question displayed earlier and says that $\chi'_t(G)$ or $\chi_t(G)$ for planar graphs G of maximum degree d and large enough girth must be at most $\tau'_t(d)$ or $\tau_t(d)$, respectively, provided t is of the correct parity. Informally, planar distance colouring becomes tree-like for high enough girth, in the right parities of distance.

Theorem 3. *For odd (even) $t \geq 1$, there exists g'_t (g_t) such that, provided d is large enough, every planar graph G of maximum degree d and girth at least g'_t (g_t) has $\chi'_t(G) \leq \tau'_t(d)$ ($\chi_t(G) \leq \tau_t(d)$). For the other parity of t, such a quantity does not exist.*

By Proposition 1, the bounds are sharp. As mentioned, g'_1 is optimally at most 3. It has also been shown that g_2 is optimally at most 7 [4]. For larger t, we have explicit bounds on g'_t and g_t, but we made little attempt to optimise them. On the contrary, we opted for short, clean proofs that

apply in general. We use the discharging method. Due to space considerations we have omitted proof details here.

We largely disregarded what happens in the "other" distance parities. Are there natural analogues to Grötzsch's Theorem in those cases?

Problem 4. *For even (odd) $t \geq 1$, what is the least $c'_t \geq 1$ ($c_t \geq 1$) such that there exists some fixed g for which the following holds? For every $\varepsilon > 0$, provided d is large enough, every planar graph G of maximum degree d and girth at least g has $\chi'_t(G) \leq (c'_t + \varepsilon)\tau'_t(d)$ ($\chi_t(G) \leq (c_t + \varepsilon)\tau_t(d)$).*

By Proposition 2, the constants c'_t and c_t are well-defined, but we do not know yet if $c'_t < \limsup_{d\to\infty} \pi'_t(d)/\tau'_t(d)$ or $c_t < \limsup_{d\to\infty} \pi_t(d)/\tau_t(d)$ in general. Grötzsch's Theorem says $c_1 \leq 3/2 < \limsup_{d\to\infty} \pi_1(d)/\tau_1(d)$, and it also implies that $c'_2 \leq 3/2 < \limsup_{d\to\infty} \pi'_2(d)/\tau'_2(d)$ [6].

Moreover, there remains the possibility that, for even t, there is a function $g'_t(d)$ (so with a dependence upon d) such that every planar graph G of maximum degree d and girth at least $g'_t(d)$ has $\chi'_t(G) \leq \tau'_t(d)$, provided d is large enough. The existence of $g'_2(d)$ and it being at most linear in d was proven in [5]. For odd t and distance-t vertex-colouring the existence of an analogous $g_t(d)$ is impossible. Moreover, $g'_t(d)$ for even t, if it exists, is at least of order $\tau'_t(d)$.

Proposition 5. *For even $t \geq 2$, there exists a planar graph G of maximum degree d and girth at least $(\tau'_t(d) - 1)/2$ such that $\chi'_t(G) > \tau'_t(d)$, provided d is large enough. For odd $t \geq 1$ and $d \geq 3$, there exists a planar graph G of maximum degree d and arbitrarily large girth such that $\chi_t(G) > \tau_t(d)$.*

The reader is referred to the full version [10] for further details.

References

[1] G. Agnarsson and M. M. Halldórsson. Coloring powers of planar graphs. *SIAM J. Discrete Math.*, 16(4):651–662, 2003.

[2] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.

[3] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.

[4] O. V. Borodin, A. N. Glebov, A. O. Ivanova, T. K. Neustroeva, and V. A. Tashkinov. Sufficient conditions for planar graphs to be 2-distance $(\Delta + 1)$-colorable. *Sib. Èlektron. Mat. Izv.*, 1:129–141, 2004.

[5] O. V. Borodin and A. O. Ivanova. Precise upper bound for the strong edge chromatic number of sparse planar graphs. *Discuss. Math. Graph Theory*, 33(4):759–770, 2013.

[6] G. J. Chang, M. Montassier, A. Pêcher, and A. Raspaud. Strong chromatic index of planar graphs with large girth. *Discuss. Math. Graph Theory*, 34(4):723–733, 2014.

[7] M. Fellows, P. Hell, and K. Seyffarth. Constructions of large planar networks with given degree and diameter. *Networks*, 32(4):275–281, 1998.

[8] H. Grötzsch. Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe*, 8:109–120, 1958/1959.

[9] F. Havet, J. van den Heuvel, C. McDiarmid, and B. Reed. List Colouring Squares of Planar Graphs. *ArXiv e-prints*, July 2008.

[10] R. J. Kang and W. van Loon. Tree-like distance colouring for planar graphs of sufficient girth. *ArXiv e-prints*, May 2018.

[11] F. Kramer and H. Kramer. Un problème de coloration des sommets d'un graphe. *C. R. Acad. Sci. Paris Sér. A-B*, 268:A46–A48, 1969.

[12] B. Laekhanukit. Parameters of two-prover-one-round game and the hardness of connectivity problems. In *Proceedings of the Twenty-fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '14, pages 1626–1643, Philadelphia, PA, USA, 2014. Society for Industrial and Applied Mathematics.

[13] E. L. Lloyd and S. Ramanathan. On the complexity of distance-2 coloring. In *Proceedings ICCI '92: Fourth International Conference on Computing and Information*, pages 71–74, May 1992.

[14] S. T. McCormick. Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem. *Math. Programming*, 26(2):153–171, 1983.

[15] M. Molloy and B. Reed. A bound on the strong chromatic index of a graph. *J. Combin. Theory Ser. B*, 69(2):103–109, 1997.

[16] S. Ramanathan and E. L. Lloyd. Scheduling algorithms for multihop radio networks. *IEEE/ACM Transactions on Networking*, 1(2):166–177, Apr 1993.

[17] D. P. Sanders and Y. Zhao. Planar graphs of maximum degree seven are class I. *J. Combin. Theory Ser. B*, 83(2):201–212, 2001.

[18] C. E. Shannon. A theorem on coloring the lines of a network. *J. Math. Physics*, 28:148–151, 1949.

[19] V. G. Vizing. Critical graphs with given chromatic class. *Diskret. Analiz No.*, 5:9–17, 1965.

[20] G. Wegner. Graphs with given diameter and a coloring problem. *Technical report, University of Dortmund*, 1977.