Strong cliques and forbidden cycles

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Abstract

Given a graph G, a *strong clique* of G is a collection of edges of G every pair of which are incident or connected by an edge in G; the *strong clique number* $\omega_2'(G)$ of G is the cardinality of a largest such collection. We study the strong clique number of graphs missing some set of cycle lengths. Among other results, we show the following: $\omega_2'(G) \leq 5\Delta(G)^2/4$ if G is triangle-free; $\omega_2'(G) \leq 3(\Delta(G)-1)$ if G is C_4-free; $\omega_2'(G) \leq \Delta(G)^2$ if G is C_5-free (where $\Delta(G)$ denotes the maximum degree of G). These bounds are attained by natural extremal examples. Our work extends and improves upon previous work of Faudree, Gyárfás, Schelp and Tuza (1990), Mahdian (2000) and Faron and Postle (2017). We are motivated by the corresponding problems for the strong chromatic index.

1 Prelude

Let $G = (V, E)$ be a graph. A *strong edge-colouring* of G is a partition of E into parts each of which induces a matching; the *strong chromatic index* $\chi_2'(G)$ of G is the least number of parts in such a partition. Although simply stated, χ_2' has proven very difficult to analyse: a conjecture of Erdős and Nešetřil [1] from the 1980s is notorious. ($\Delta(G)$ denotes the maximum degree of G.)

Conjecture 1 (Erdős and Nešetřil [1])**.** *For any graph G, $\chi_2'(G) \leq 5\Delta(G)^2/4$.*

If true, this bound would be sharp for $\Delta(G)$ even, by considering a 5-cycle and substituting each of its vertices with a copy of a stable set (of $\Delta(G)/2$ vertices). Molloy and Reed [8] proved the existence of some absolute constant $\varepsilon > 0$ such that $\chi_2'(G) \leq (2-\varepsilon)\Delta(G)^2$ for any graph G. Unfortunately, despite much effort, there is no explicit lower estimate on ε greater than, say, 0.001 (although slightly better results are known when $\Delta(G)$ is large).

The problem is still difficult if we restrict attention to a (non-trivial) class defined by some forbidden set of subgraphs, say, cycles. The following has also been open since the 1980s.

Conjecture 2 (Faudree, Gyárfás, Schelp and Tuza [3])**.** *For any bipartite graph G, $\chi_2'(G) \leq \Delta(G)^2$.*

Balanced complete bipartite graphs meet the conjectured bound. A more general bound has been conjectured: perhaps it suffices to forbid just one odd cycle length rather than all of them.

Conjecture 3 (Mahdian [7])**.** *For any C_5-free graph G, $\chi_2'(G) \leq \Delta(G)^2$.*

By contrast, the best understood classes are when the forbidden set includes some bipartite graph. In particular, Mahdian [7] showed the following for C_4. This was subsequently extended to a similar statement with C_4 replaced by any bipartite H [9]. See [5] for an explicit derivation with $H = C_{2k}$.

Theorem 4 (Mahdian [7])**.** *Fix $\varepsilon > 0$. If G is a C_4-free graph, then, provided $\Delta(G)$ is large enough, $\chi_2'(G) \leq (2+\varepsilon)\Delta(G)^2/\log\Delta(G)$. This is sharp up to the constant multiplicative factor.*

Despite this result, the graphs of small maximum degree remain a potential obstruction to the outright confirmation of Erdős and Nešetřil's conjecture for the class of all, say, C_4-free graphs.

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2 Main results

From the above rather frustrating state of affairs, there is justification to the pursuit of a somewhat weaker parameter than χ'_2, even for restricted graph classes. In particular, a *strong clique* of G is a collection of edges of G every pair of which are incident or connected by an edge in G; the *strong clique number* $\omega'_2(G)$ of G is the cardinality of a largest such collection. Since the edges of a strong clique must be assigned to different parts in a strong edge-colouring, $\omega'_2(G) \leq \chi'_2(G)$ always. The originators provided the following supporting evidence for Conjecture 2.

Theorem 5 (Faudree, Gyárfás, Schelp and Tuza [4])**.** *For any bipartite graph G, $\omega'_2(G) \leq \Delta(G)^2$.*

Our main results are as follows.

Theorem 6. *Let G be a graph with maximum degree $\Delta(G) \geq 4$. Then the following hold:*

1. *$\omega'_2(G) \leq 5\Delta(G)^2/4$ if G is triangle-free;*
2. *$\omega'_2(G) \leq 3(\Delta(G) - 1)$ if G is C_4-free; and*
3. *$\omega'_2(G) \leq \Delta(G)^2$ if G is C_5-free.*

Although it would "just" be a consequence of a special case of Conjecture 1, Theorem 6.1 is best possible for even $\Delta(G)$ due to the blown-up 5-cycles. Theorem 6.2 is sharp by considering a triangle to each vertex of which is added $\Delta(G) - 2$ pendant edges. This bound (slightly) improves upon a bound announced in Mahdian's master's thesis [6]. Observe how Theorem 6.2 compares with Theorem 4. Theorem 6.3 is sharp for the balanced complete bipartite graphs. It simultaneously extends Theorem 5 and another result of Mahdian [7, Thm. 15] and provides (further) evidence towards Conjecture 3.

In the full version (in preparation), we also establish bounds for $\omega'_2(G)$ when G is free of other (sets of) cycle lengths. Those bounds provide a contrast between the even and odd parities of cycle lengths. We also consider a more challenging parameter. A *distance-t edge-clique* of G is a collection of edges of G that form a clique in the t-th power $L(G)^t$ of the line graph $L(G)$ of G; the *distance-t edge-clique number* of G is the cardinality of a largest distance-t edge-clique of G. The proposal of such parameters was already hinted at by Erdős [1, p. 81]. We study the effect on the distance-t edge-clique number of excluding a modest fixed set of cycle lengths. Most of our bounds are (nearly) extremal, but we omit them here for the sake of brevity and clarity.

3 Methodology

Recently, Faron and Postle [2] strengthened Theorem 5 by use of the *Ore-degree* $\sigma(G)$ of G, which is defined as the largest over all edges of G of the sum of the two endpoint degrees.

Theorem 7 (Faron and Postle [2])**.** *For any bipartite graph G, $\omega'_2(G) \leq \sigma(G)^2/4$.*

Since the sum of two degrees in G is always at most $2\Delta(G)$, this directly implies Theorem 5. Theorem 7 follows from a slightly more technical version. For a sub(multi)graph H of a (multi)graph G, the Ore-degree $\sigma_G(H)$ of H in G is $\max_{xy \in E(H)} (\deg_G(x) + \deg_G(y))$, where $\deg_G(x)$ denotes the degree of x in G.

Lemma 8 (Faron and Postle [2])**.** *If G is a bipartite multigraph and H is a sub(multi)graph of G such that $E(H)$ is a clique in $L(G)^2$, then $|E(H)| \leq \Delta(H) \cdot (\sigma_G(H) - \Delta(H)) \leq \sigma_G(H)^2/4$.*

We have further extended Theorem 7 (and Lemma 8), and have Theorem 6.3 as a corollary.

Theorem 9. *For any C_5-free graph G, $\omega_2'(G) \leq \sigma(G)^2/4$.*

We omit the proof of Theorem 9, but show how it, or rather Lemma 8, yields Theorem 6.1.

Proof of Theorem 6.1. Let H be a subgraph of G whose edges form a maximum clique in $L(G)^2$. From now on we call H and its edges *blue*. Let $v \in V(G)$ be of maximal blue degree s. Let V_T denote the set of vertices that are incident to an edge of H that is not incident to $N[v]$. Let $G_T = (V_T, E_T)$ be the graph induced by V_T and $H_T = (V_T, E_T \cap E(H))$ the blue subgraph of G_T.

Let $C_1, C_2, \ldots$ denote the connected components of H_T. Let pq be an edge in component C_i. For all $x \in N_H(v)$, the blue edges xv and pq must be within distance 2. They are not incident, so either $xp \in E(G)$ or $xq \in E(G)$. By triangle-freeness, we cannot have both $xp, xq \in E(G)$. It follows that *pq partitions* $N_H(v)$ into $A_i := N_G(p) \cap N_H(v)$ and $\overline{A_i} := N_G(q) \cap N_H(v)$. We will call $(A_i, \overline{A_i})$ the partition induced by pq. Now suppose C_i contains another edge qr which is incident to pq. Then by triangle-freeness, qr induces the same partition. It follows inductively that all edges in C_i induce the *same* partition $(A_i, \overline{A_i})$ of $N_H(v)$.

Let $C_1, \ldots, C_k$ be the components that induce the trivial partition $(\emptyset, N_H(v))$ (if they exist). Let $M = |C_1| + \cdots + |C_k|$ denote the number of (blue) edges that are in these 'trivial' components. On the other hand, let $G_B := G[\bigcup_{i \geq k+1} V(C_i)]$ and $H_B := H[\bigcup_{i \geq k+1} V(C_i)]$ be the graphs induced by the remaining 'nontrivial' components.

Claim 10. $M \leq (\Delta - s)\Delta$.

Claim 11. $\sigma_{G_B}(H_B) \leq 2\Delta - s - M/\Delta$.

Claim 12. *G_B is bipartite.*

We postpone the proofs of these claims. Note that $E(H_B)$ is a clique not only in $L(G)^2$ but also in $L(G_B)^2$. So by Claim 12, we may apply Lemma 8 and then Claim 11, yielding

$$|E(H_B)| \leq \frac{\sigma_{G_B}(H_B)^2}{4} \leq \frac{(2\Delta - s - M/\Delta)^2}{4}.$$

It follows that $\omega(L(G)^2)$ is at most

$$\begin{aligned} |E(H)| &= \#\{e \in E(H) \mid e \text{ incident to } N_G(v)\} + |E(H_T)| \leq s \cdot |N_G(v)| + M + |E(H_B)| \\ &\leq s\Delta + M + \frac{(2\Delta - s - M/\Delta)^2}{4} = \Delta^2 + \frac{1}{4}\left(s + \frac{M}{\Delta}\right)^2 \leq \frac{5}{4}\Delta^2, \end{aligned}$$

where we used Claim 10 in the last line. This concludes the proof, conditioned on Claims 10–12.

Given the i-th component C_i, let X_i respectively Y_i denote the set of vertices in C_i whose neighbourhood in $N_H(v)$ is A_i respectively $\overline{A_i}$. Note that X_i is complete to A_i and Y_i is complete to $\overline{A_i}$. Furthermore, the bipartite subgraph of H induced by C_i has parts X_i and Y_i.

Proof of Claim 10. If C_i is a trivial component ($1 \leq i \leq k$) then Y_i is complete to $\overline{A_i} = N_H(v)$. Therefore $|\bigcup_{1 \leq i \leq k} Y_i| \leq \Delta$, and for the same reason all $y \in \bigcup_{1 \leq i \leq k} Y_i$ satisfy $|N_{H_T}(y)| \leq \Delta - s$. So $M \leq |N_{H_T}(\bigcup_{1 \leq i \leq k} Y_i)| \leq \Delta(\Delta - s)$. ♢

Proof of Claim 11. Let $e = pq \in E(H_B)$. Then for all $x \in N_H(v)$, x must be adjacent to either p or q. So there are $|N_H(v)| = s$ edges between $\{p, q\}$ and $N_H(v)$. Also, pq must be at distance 2 of every of the M edges induced by the trivial components. So there are at least $\frac{M}{\Delta}$ edges between $\{p, q\}$ and the trivial components. So at least $s + M/\Delta$ edges incident to $\{p, q\}$ are not in G_B. It follows that $\sigma_{G_B}(e) = d_{G_B}(p) + d_{G_B}(q) \leq 2\Delta - s - M/\Delta$. $\diamondsuit$

Proof of Claim 12. Suppose there are two different nontrivial components, C_i and C_j. We will first show that we may then assume that either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Indeed: if either $\overline{A_j} \subseteq A_i$ or $A_i \subseteq \overline{A_j}$, then after interchanging X_j and Y_j (and thus interchanging A_j and $\overline{A_j}$), we obtain $A_j \subseteq A_i$ or $A_i \subseteq A_i$. So we may assume for a contradiction that none of $A_i \subseteq A_j, A_i \subseteq \overline{A_j}, A_j \subseteq A_i, \overline{A_j} \subseteq A_i$ holds. But then there exist $a \in A_i \cap \overline{A_j}, b \in A_i \cap A_j, c \in \overline{A_i} \cap A_j$ and $d \in \overline{A_i} \cap \overline{A_j}$. Furthermore, because each component contains at least one blue edge, there are blue edges $(x_i, y_i) \in (X_i \times Y_i)$ and $(x_j, y_j) \in (X_j \times Y_j)$ that have to be connected by an edge in order to have them within distance 2. If $x_i x_j$ is an edge, then $x_i x_j b$ forms a triangle. Similarly, if $x_i y_j$, $y_i y_j$ or $x_j y_i$ is an edge then $x_i y_j a$, $y_i y_j d$ or $x_j y_i c$ is a triangle, respectively. Contradiction.

It follows that we can reorder the components by inclusion, so that $A_{k+1} \subseteq A_{k+2} \subseteq \cdots$. Now we are ready to show that G_B is bipartite, on parts $X := \bigcup_{i \geq k+1} X_i$ and $Y := \bigcup_{i \geq k+1} Y_i$. Suppose X is not a stable set. Then there are $x_1, x_2 \in X$ that form an edge, where $x_1 \in X_i$ and $x_2 \in X_j$ for some $i \leq j$. Since $\emptyset \neq A_i \subseteq A_j$, there must be a triangle in $x_1 x_2 A_i$. Contradiction. Similarly, suppose Y is not a stable set. Then there are $y_1, y_2 \in Y$ that form an edge, where $y_1 \in Y_i$ and $y_2 \in Y_j$ for some $i \leq j$. Since $\emptyset \neq \overline{A_j} \subseteq \overline{A_i}$, there must be a triangle in $y_1 y_2 \overline{A_j}$. Contradiction. $\diamondsuit$

This completes the proof of Theorem 6.1. $\square$

Theorem 6.2 follows from a case analysis which we have chosen to omit from this extended abstract.

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