Cuts in matchings of 3-edge-connected cubic graphs

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Abstract

We discuss relations between several known (some false, some open) conjectures on 3-edge-connected, cubic graphs and how they all fit into the same framework related to cuts in matchings. We then provide a construction of 3-edge-connected digraphs satisfying the property that for every even subgraph E, the graph obtained by contracting the edges of E is not strongly connected. This disproves a recent conjecture of Hochstättler [A flow theory for the dichromatic number. *European Journal of Combinatorics*, 66, 160–167, 2017]. Moreover, we provide experimental evidence for these conjectures and discuss on tools that might be helpful to search for counterexamples.

In an attempt to solve the Four Color Problem, Tait conjectured the following:

Conjecture 1 (Tait 1884)**.** *Every 3-edge-connected, cubic, planar graph contains a Hamiltonian cycle.*

The first counterexample to Conjecture 1 was given by Tutte [10] and is a graph on 46 vertices. Several smaller counterexamples on 38 vertices were later found by Holton and McKay, who also proved that there is no counterexample with less than 38 vertices [6]. One can observe that all known counterexamples to Tait's conjecture have odd cycles. Maybe this is essential:

Conjecture 2 (Barnette 1969 [1])**.** *Every 3-edge-connected, cubic, planar, bipartite graph contains a Hamiltonian cycle.*

In general Conjecture 2 remains open. It was shown to be true for graphs with at most 66 vertices [5]. A few years later a stronger conjecture was proposed:

Conjecture 3 (Tutte 1971 [11])**.** *Every 3-edge-connected, cubic, bipartite graph contains a Hamiltonian cycle.*

Conjecture 3 was disproved by Horton [4]. The smallest known counterexample has 50 vertices and was discovered independently by Georges [2] and Kelmans [7]. Moreover, in [7] it is claimed (but no reference given) that Lomonosov and Kelmans proved Conjecture 3 for graphs on at most 30 vertices. We verified this claim by computer by showing that the conjecture holds for graphs on up to 38 vertices.

Observe that a cubic graph $G = (V, E)$ contains a Hamiltonian cycle C, if and only if the complement $E \setminus C$ is a perfect matching containing no edge-cut. Therefore, in Conjectures 1, 2, 3 the property *contains a Hamiltonian cycle* can be replaced equivalently by *contains a perfect matching without cut*. Thus we can reformulate all of the above conjectures as *Every 3-edge-connected cubic (bipartite) and/or (planar) graph has a perfect matching without cut.*

Directed graphs

Let us now go to the directed setting. The *digirth* of a digraph $D = (V, A)$ is the length of a shortest directed cycle. A digraph is called *oriented graph* if it is of digirth at least 3. A set of vertices $V' \subseteq V$ is acyclic in D if the digraph induced by V' contains no directed cycle. Neumann-Lara stated the following:

Conjecture 4 (Neumann-Lara 1985 [9]). *Every planar oriented graph can be vertex-partitioned into two acyclic sets.*

Conjecture 4 remains open in general, but was recently proved for graphs with digirth at least 4 [8]. Here we give the first computational evidence for it by showing that it is valid for all planar graphs on at most 26 vertices.

Given a (partially) directed graph $D = (V, A)$, for $E \subseteq A$, let D/E denote the graph obtained from D by contracting the edges of E. An *even* subgraph E of a digraph $D = (V, A)$ is a subset $E \subseteq A$ that is an edge-disjoint union of cycles of D (the cycles are not necessarily directed). For a planar oriented graph D, we would like to make several easy observations:

- Partitioning the vertex set of in two acyclic sets corresponds to finding an edge-cut whose removal leaves an acyclic directed graph.
- Since D has digirth at least 3 it can be assumed to be simple. Thus, the planar dual digraph D^* is 3-edge-connected.
- The cut in D corresponds to an even subgraph in the planar dual digraph D^*, whereas an acyclic directed planar graph is dual to a strongly connected planar graph. On the other hand, deleting an edge in D corresponds to contracting it in D^*.

Thus, Conjecture 4 can be reformulated as: *every 3-edge-connected, planar digraph $D = (V, A)$ contains an even subgraph $E \subseteq A$ such that D/E is strongly connected.*

Following the observations discussed above, Hochstättler proposed a generalization of Conjecture 4:

Conjecture 5 (Hochstättler 2017 [3]). *In every 3-edge-connected digraph $D = (V, A)$ there exists an even subgraph $E \subseteq A$ such that D/E is strongly connected.*

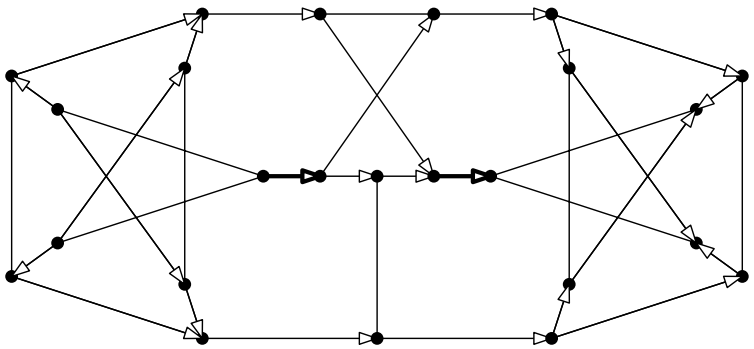


Figure 1: Smallest oriented cubic, 3-edge-connected, partially directed graph where each perfect matching contains a cut (the undirected edges can be oriented arbitrarily)

In this paper we construct a counterexample for Conjecture 5. In order to do so, we show that Conjectures 4 and 5 are equivalent to the following:

Conjecture IV (Neumann-Lara). *Every 3-edge-connected, cubic, planar digraph contains a perfect matching without directed cut.*

Conjecture V (Hochstättler)**.** *Every 3-edge-connected, cubic digraph contains a perfect matching without directed cut.*

A counterexample to Conjecture V, that is a directed graph where each perfect maching has a directed cut, is given in Figure 1. In fact, by computer search we verified that the partially oriented graph of Figure 1 is the the smallest counterexample to Conjecture V.

As of Conjecture IV, our experimental results show that it holds for planar cubic 3-edge-connected graphs on at most 48 vertices. If moreover the graphs have cyclic connectivity at least 4, we verified it for graphs on 50 vertices too. Therefore, by planar duality, we conclude that Conjecture 4 holds for planar graphs on at most 26 vertices and for 4-vertex-connected planar graphs on at most 27 vertices.

Note that from the previous discussion a new question follows naturally and, after checking it on some sets of graphs, we propose the following:

Conjecture 6. *Every 3-edge-connected, cubic, bipartite digraph contains a perfect matching without directed cut.*

See Figure 2 for an illustration of the relations between all the conjectures discussed in this manuscript.

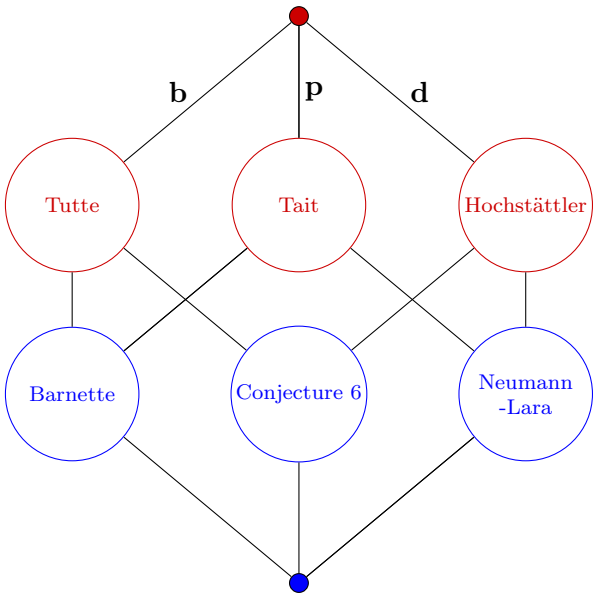


Figure 2: Conjectures of the form "*Every 3-edge-connected, cubic,* ***b****ipartite,* ***p****lanar,* ***d****irected graph contains a perfect matching without (directed) cut*". The upper four are false, the lower four are open.

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