On dense bipartite induced subgraphs[1]

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Abstract

We conjecture that every triangle-free graph of minimum degree d contains a bipartite induced subgraph of minimum degree $\Omega(\log d)$ as $d \to \infty$. In this extended abstract, we present some results on this conjecture. In particular we prove that the conjecture holds for (nearly) regular graphs, and also for graphs with high average degree (as a function of n). In the general case we also prove a "half" version of the statement, where we only ask one of the two parts of the bipartition to induce a stable set.

1 Introduction

A well known observation is that in a maximum cut (i.e. a bipartition maximising the number of edges between the two parts), any vertex has at least half of its neighbours on the other side of the bipartition. In particular, this implies the following folklore result.

Proposition 1. *For any integer d, any graph with minimum degree at least d contains a bipartite subgraph of minimum degree at least $d/2$.*

This basic result is very useful, and it is natural to wonder how we can make a stricter conclusion where the high degree bipartite subgraph we seek is *induced* (i.e. the two partite sets are stable sets in the original graph). Note that in a complete graph, any bipartite induced subgraph contains at most two vertices, so then we need to exclude large cliques. We conjecture the following.

Conjecture 2. *There are functions $f(d)$ and $g(d)$ satisfying $f(d) \to \infty$ and $g(d) \to \infty$ as $d \to \infty$ such that any graph with minimum degree at least d contains a complete subgraph on $f(d)$ vertices or a bipartite induced subgraph with minimum degree at least $g(d)$.*

Our original motivation for Conjecture 2 is a list colouring problem; see the full version of the paper [4] for more details. A tantalising special case of this conjecture concerns excluded cliques of size 3, for which we propose something more precise.

Conjecture 3. *There is a constant $C > 0$ such that any triangle-free graph with minimum degree at least d contains a bipartite induced subgraph of minimum degree at least $C \log d$.*

We offer some partial progress, especially towards Conjecture 3. To that end we have found it quite useful that, to find a dense bipartite induced subgraph in a graph of given minimum degree, it suffices to find a good proper colouring.

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Theorem 4. *Any graph with chromatic number at most k and minimum degree d has a bipartite induced subgraph of minimum degree at least $d/2k$.*

We show this as corollary to a more general result below, Theorem 5, which in turn follows from the even more general Lemma 6. In Section 3, we illustrate how, together with classic results about colourings and stable sets in triangle-free graphs, Theorem 4 yields special cases of Conjecture 3: in particular, if the triangle-free graph is nearly regular (Theorem 8) or if it has sufficiently large minimum degree with respect to the number of vertices (Theorem 9). Theorem 4 also implies that Conjecture 2 is true for regular or very high minimum degree graphs (Theorems 10 and 11).

By a related but different method, we prove a weaker form of Conjecture 3 where edges are permitted for one of the two parts of the sought bipartite subgraph (Theorem 13). We also show that our result and Conjecture 3 (if true) would be sharp up to a multiplicative constant (Theorem 14).

2 Fractional colouring

Since every graph of average degree d contains a subgraph of minimum degree at least $d/2$, and since we do not worry ourselves too much about constant factors, we can work with average or minimum degree interchangeably.

Given a graph $G = (V, E)$, let us say that a probability distribution $\mathcal{S}$ over the stable sets of G satisfies property Q^*_r if $\mathbb{P}(v \in \mathbf{S}) \geq r$ for every vertex $v \in V$ and $\mathbf{S}$ a random stable set taken according to $\mathcal{S}$. Recall that the *fractional chromatic number* of G may be defined as the smallest k such that there is a probability distribution over the stable sets of G that satisfies property $\mathrm{Q}^*_{1/k}$.

Theorem 5. *Any graph with fractional chromatic number at most k and minimum degree d has a bipartite induced subgraph of average degree at least d/k.*

By considering the distribution that takes a colour class of a proper colouring uniformly at random, it is easily seen that the fractional is bounded by the usual chromatic number, and so this immediately implies Theorem 4.

Indeed, we will only need something slightly weaker than the notion of fractional colouring. More precisely, we say that a probability distribution $\mathcal{S}$ over the stable sets of G satisfies property P^*_q if $\mathbb{E}(|N(v) \cap \mathbf{S}|) \geq q$ for every vertex $v \in G$ and $\mathbf{S}$ a random stable set taken according to $\mathcal{S}$. Such a distribution guarantees a dense bipartite induced subgraph as follows.

Lemma 6. *For any graph and any q, if there exists a probability distribution $\mathcal{S}$ over the stable sets of the graph that satisfies property P^*_q, then the graph has a bipartite induced subgraph of average degree at least q.*

Proof. Let $G = (V, E)$ be a graph and let $\mathcal{S}$ be a probability distribution over the stable sets of G with property P^*_q for some q. Let $\mathbf{S_1}$ and $\mathbf{S_2}$ be two stable sets taken independently at random according to $\mathcal{S}$. By property P^*_q and linearity of expectation, for any fixed $S \subseteq V$, the expected number of edges with one endpoint in $\mathbf{S_1}$ and the other in S (or with one endpoint in $\mathbf{S_2}$ and the other in S) is at least $q|S|$. Therefore, by independence, the number of edges in the subgraph induced by the set $\mathbf{S_1} \cup \mathbf{S_2}$ satisfies $\mathbb{E}(|E(G[\mathbf{S_1} \cup \mathbf{S_2}])|) \geq q\,\mathbb{E}(|\mathbf{S_2}|)$. And by a symmetric argument, we also have $\mathbb{E}(|E(G[\mathbf{S_1} \cup \mathbf{S_2}])|) \geq q\,\mathbb{E}(|\mathbf{S_1}|)$, and thus $\mathbb{E}(|E(G[\mathbf{S_1} \cup \mathbf{S_2}])|) \geq \frac{q}{2}(\,\mathbb{E}(|\mathbf{S_1}|) + \mathbb{E}(|\mathbf{S_2}|))$. By linearity of expectation,

$$\mathbb{E}\left(|E(G[\mathbf{S_1} \cup \mathbf{S_2}])| - \tfrac{q}{2}(|\mathbf{S_1}| + |\mathbf{S_2}|)\right) \geq 0.$$

Therefore there are two stable sets S_1 and S_2 of G with at least $\frac{q}{2}(|S_1|+|S_2|)$ edges in the subgraph induced by $S_1 \cup S_2$. Discarding the vertices of $S_1 \cap S_2$ (if any) yields a bipartite induced subgraph of average degree at least q. □

Proof of Theorem 5. By the assumption, the graph has a distribution $\mathcal{S}$ over the stable sets with property $\mathrm{Q}^*_{1/k}$. Let $\mathbf{S}$ be a random stable set from this distribution. For each vertex v of the graph, each neighbour of v is in $\mathbf{S}$ with probability at least $1/k$. Thus, since the graph has minimum degree d, it follows from linearity of expectation that $\mathbb{E}(|N(v) \cap \mathbf{S}|) \geq d/k$. So $\mathcal{S}$ also has property $\mathrm{P}^*_{d/k}$ and the result follows from Lemma 6. □

3 Some consequences of Theorem 5

First let us note that, by an induction on the number of vertices, we may assume in Conjecture 3 that any proper subgraph of the triangle-free graph has minimum degree less than d. (Note that the base case is implied by Mantel's theorem.) So the graph may be assumed to be d-degenerate. By Theorem 5, Conjecture 3 is thus implied by a recent conjecture of Harris.

Conjecture 7 (Harris [5])**.** *There is some $C > 0$ such that the fractional chromatic number of any d-degenerate triangle-free graph is at most $Cd/\log d$.*

Next we observe that, due to classic results on the chromatic number of triangle-free graphs [6, 8] and on off-diagonal Ramsey numbers [1], it directly follows from Theorem 5 that we may also assume in Conjecture 3 that the maximum degree of the graph is large with respect to d and that the minimum degree d is not too large with respect to the number of vertices.

Theorem 8. *There is a constant $C > 0$ such that any triangle-free graph with minimum degree d and maximum degree Δ contains a bipartite induced subgraph of average degree at least $\frac{d}{C\Delta}\log\Delta$.*

Theorem 9. *There is a constant C_1 such that any triangle-free graph on n vertices with average degree $d \geq Cn^{2/3}\sqrt{\log n}$ contains a bipartite induced subgraph of average degree at least $\frac{CC_1}{2}\log n$ for all n large enough.*

The following result has the same proof as that of Theorem 8, but uses instead the fact that K_r-free graphs of maximum degree Δ have chromatic number at most $200r\Delta\log\log\Delta/\log\Delta$. This was recently proved by Molloy [8], improving an earlier result of Johansson [7].

Theorem 10. *There is some constant $C > 0$ such that, for every $r \geq 4$, any K_r-free graph with minimum degree d and maximum degree Δ contains a bipartite induced subgraph of average degree at least $\frac{d\log\Delta}{Cr\Delta\log\log\Delta}$.*

The next result is shown in the same way as Theorem 9, but using a slightly more general bound [1] for the off-diagonal Ramsey numbers: $R(r,\ell) = O(\ell^{r-1}/(\log\ell)^{r-2})$ as $\ell \to \infty$.

Theorem 11. *For each $r \geq 3$, any K_r-free graph G on n vertices with average degree $d = \Omega(n^{1-1/r}(\log n)^{1/(r-1)})$ contains a bipartite induced subgraph of average degree $\Omega(\log n)$ as $n \to \infty$.*

4 Dense semi-bipartite subgraphs

Instead of a dense bipartite *induced subgraph*, we might be happy with a dense bipartite *subgraph* where we only demand (at least) one of the parts induces a stable set. Given a graph $G = (V, E)$, let us call an induced subgraph $G' = (V', E')$ of G *semi-bipartite* if it admits a partition $V' = V_1 \cup V_2$ such that V_1 is a stable set of G, and define the average degree of G' with respect to the semi-bipartition as the average degree of the bipartite subgraph $G[V_1, V_2]$ between V_1 and V_2 (and so we ignore any edges in V_2).

Borrowing ideas from a proof of Ajtai, Komlós and Szemerédi [2] (see also the book of Alon and Spencer [3, p. 321–2]), we can prove the following result.

Lemma 12. *Let G be a triangle-free graph of minimum degree $d \geq 1$, and $\mathbf{S}$ a random stable set of G chosen uniformly. Then $\mathbb{E}(\sum_{v \in \mathbf{S}} d(v)) \geq \frac{1}{4} \sum_{v \in G} \log d(v)$.*

From this, it is not difficult to derive the following weak version of Conjecture 3.

Theorem 13. *Any triangle-free graph of minimum degree $d \geq 1$ contains a semi-bipartite induced subgraph of average degree at least $\frac{1}{2} \log d$.*

Using a probabilistic construction, we prove that this is sharp up to a constant factor.

Theorem 14. *There are constants $C, C', C'' > 0$ such that for every large enough n there is a triangle-free graph on n vertices with all degrees between $C' n^{1/3}$ and $C'' n^{1/3}$ that contains no semi-bipartite induced subgraph of average degree at least $C \log n$.*

Observe that this also implies that Conjecture 3, if true, would be sharp up to a constant factor.

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