Graceful difference labelings of disjoint directed cycles

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Abstract

A graceful difference labeling of a directed graph G with vertex set V is a bijection $f : V \to \{1, \ldots, |V|\}$ such that, when each arc uv is assigned the difference label $f(v) - f(u)$, the resulting arc labels are distinct. We are interested in the case of a disjoint unions of directed cycles. When $G = n\overrightarrow{\mathbf{C_3}}$, i.e. a collection of n directed triangles $n \geq 1$, we show that G has a graceful difference labeling if and only if $n \neq 1$.

1 Introduction

A graph labeling is the assignment of integers, to the vertices or edges, or both, of a graph, subject to certain conditions. Among all variations, the most popular and studied graph labelings are the β-valuations introduced by Rosa in 1966 and later called *gracelful labelings* by Golomb [1]. Formally, given a graph G with vertex set V and q edges, a graceful labeling of G is an injection $f : V \to \{0, 1, \ldots, q\}$ such that, when each edge uv is assigned the label $|f(v) - f(u)|$, the resulting edge labels are distinct. The famous Ringel-Kotzig conjecture, also known as the graceful labeling conjecture, hypothesizes that all trees are graceful.

For a directed graph with vertex set V and q edges, a graceful labeling of G is an injection $f : V \to \{0, 1, \ldots, q\}$ such that, when each arc (i.e., directed edge) uv is assigned the label $(f(v) - f(u))$ $(mod\ q + 1)$, the resulting arc labels are distinct. As mentioned in [1] and [2], most results and conjectures on graceful labelings of directed graphs concern directed cycles, the disjoint union of directed cycles, and the union of directed cycles with one common vertex or one common arc. In particular, it is proved that $n\overrightarrow{\mathbf{C_3}}$, the disjoint union of n copies of the directed cycle with three vertices, has a graceful labeling only if n is even. However, it is not known whether this necessary condition is also sufficient.

Here, we study *graceful difference labelings* of directed graphs, which are defined as follows. A graceful difference labeling (gdl for short) of a directed graph $G = (V, A)$ is a bijection $f : V \to \{1, \ldots, |V|\}$ such that, when each arc uv is assigned the *difference label* $f(v) - f(u)$, the resulting arc labels are distinct. We show that $n\overrightarrow{\mathbf{C_3}}$ has a graceful difference labeling if and only if $n \neq 1$.

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2 Proof's outline

- $n = 1$: $\overrightarrow{\mathbf{C_3}}$ has no gdl since all bijections $f : V \to \{1, 2, 3\}$ induce two difference labels equal to 1, or two equal to -1.

- $n > 1$: We prove the result by induction on n.

  The initialization step consists of a graceful difference labeling with some specific structure for $2 \leq n \leq 9$.

  For $n \geq 10$, we assume the result is true for less than n directed triangles. Let t and r be two integers such that $-4 \leq r \leq 2$ and $n = 7t + r$. We thus have $t \geq 2$. We show how to construct a gdl for $n\overrightarrow{\mathbf{C_3}}$ given a gdl for $t\overrightarrow{\mathbf{C_3}}$.

  We have to add $n - t$ directed triangles to $t\overrightarrow{\mathbf{C_3}}$. Let $\theta = \lceil \frac{n-t}{2} \rceil = 3t + \lceil \frac{r}{2} \rceil$.

  It follows that $n - t = 2\theta$ if r is even, and $n - t = 2\theta - 1$ if r is odd. We prove the result by considering the 4 cases A,B,C,D defined in Table 1.

$n-t$	r	θ	Case
2θ	-4	$3t-2$	A
	-2	$3t-1$	
	0	$3t$	
	2	$3t+1$	B
$2\theta-1$	-3	$3t-1$	C
	-1	$3t$	
	1	$3t+1$	D

Table 1: Four different cases

We summarize here below the main arguments used for Case A, that is for $n = 2\theta + t$ and $\theta \in \{3t - 2, 3t - 1, 3t\}$.

The gdl is built as follows: Consider 2θ directed triangles $T_1, \ldots, T_{2\theta}$, every T_i having $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$ as vertex set and $\{v_{3i-2}v_{3i-1},\ v_{3i-1}v_{3i},\ v_{3i}v_{3i-2}\}$ as arc set. Consider the vertex labels $f(v_i)$ for $T_1, \ldots, T_{2\theta}$ shown in Table 2.

Triangle T_i	$f(v_{3i-2})$	$f(v_{3i-1})$	$f(v_{3i})$
T_1	1	$2\theta+1$	$6\theta+3t-3$
T_2	2	$6\theta+3t$	$4\theta+3t$
T_3	3	$6\theta+3t-1$	$2\theta+2$
T_4	4	$4\theta+3t-1$	$6\theta+3t-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
T_{2k-1}	$2k-1$	$2\theta+k$	$6\theta+3t-2k+2$
T_{2k}	$2k$	$6\theta+3t-2k+1$	$4\theta+3t-k+1$
$k = 3, \ldots, \theta$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2: The labeling of $T_1, \ldots, T_{2\theta}$ for case A.

Also, let f' be a gdl for $t\overrightarrow{\mathbf{C_3}}$ and let $f(v_i) = f'(v_i) + 3\theta$ for $i = 6\theta+1, \ldots, 6\theta+3t$. We show how to modify this initial labeling f of the vertex set of $n\overrightarrow{\mathbf{C_3}}$ to obtain a gdl. Since the proof is quite technical, we only present its main steps.

For each T_i, we define its *small* difference label as the minimum among $|f(v_{3i-1}) - f(v_{3i-2})|$, $|f(v_{3i}) - f(v_{3i-1})|$, and $|f(v_{3i-2}) - f(v_{3i})|$. Similarly, the *big* difference label of T_i is the maximum of these three values, and the *medium* one is the third value on T_i. Table 3 gives the small, medium and big difference labels of $T_1, \ldots, T_{2\theta}$. By considering two dummy directed triangles D_1 and D_2, we group the triangles into $\theta + 1$ pairs $\pi_0, \ldots, \pi_\theta$, as shown in Table 3.

Pair	Triangle	Small-dl	Medium-dl	Big-dl
$\pi_0 = (T_1, T_2)$	T_1	2θ	$4\theta + 3t - 4$	$-(6\theta + 3t - 4)$
	T_2	-2θ	$-(4\theta + 3t - 2)$	$6\theta + 3t - 2$
$\pi_1 = (T_3, T_4)$	T_3	$-(2\theta - 1)$	$-(4\theta + 3t - 3)$	$6\theta + 3t - 4$
	T_4	$2\theta - 1$	$4\theta + 3t - 5$	$-(6\theta + 3t - 6)$
$\pi_2 = (D_1, T_5)$	D_1	$-(2\theta - 2)$	$-(4\theta + 3t - 5)$	$6\theta + 3t - 7$
	T_5	$2\theta - 2$	$4\theta + 3t - 7$	$-(6\theta + 3t - 9)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi_k = (T_{2k}, T_{2k+1})$	T_{2k}	$-(2\theta - k)$	$-(4\theta + 3t - 3k + 1)$	$6\theta + 3t - 4k + 1$
$k = 3, \ldots, \theta - 1$	T_{2k+1}	$2\theta - k$	$4\theta + 3t - 3k - 1$	$-(6\theta + 3t - 4k - 1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi_\theta = (T_{2\theta}, D_2)$	$T_{2\theta}$	$-\theta$	$-(\theta + 3t + 1)$	$2\theta + 3t + 1$
	D_2	θ	$\theta + 3t - 1$	$-(2\theta + 3t - 1)$

Table 3: The difference labels of the arcs of $T_1, \ldots, T_{2\theta}, D_1, D_2$ for case A.

Then we use the following properties: first, in each π_i its two small difference labels are opposite, from π_2 to π_θ the big difference labels have the same parity, and for the medium difference labels, their parities alternate between successive pairs; second, the largest absolute value for a small difference label is less than the smallest absolute value for a medium difference label; third, the two sequences of the absolute values of the medium difference labels and big difference labels are decreasing from π_1 to π_θ.

The above properties show that some medium difference labels and big difference labels can still be equal. This yields to a conflict graph: the vertices are the pairs π_i's and the graph is oriented according to the equalities between the medium and big difference labels. We prove that this graph is an oriented forest of height at most two.

By considering each connected component of the conflict graph, permuting the labels inside the two triangles of some well chosen π_i's we obtain a gdl.

The graceful difference labelings for cases B,C, and D are obtained in a similar way.

References

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