A bandwidth theorem for approximate decompositions

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Abstract

We provide a degree condition on a regular n-vertex graph G which ensures the existence of a near optimal packing of any family $\mathcal{H}$ of bounded degree n-vertex k-chromatic separable graphs into G. In general, this condition is best possible.

Here a graph is separable if it has a sublinear separator whose removal results in a set of components of sublinear size. Equivalently, this condition can be replaced by that of having small bandwidth. Thus our result can be viewed as a version of the bandwidth theorem of Böttcher, Schacht and Taraz in the setting of approximate decompositions.

As an example, our result implies that if $\mathcal{H}$ is a family of bounded degree n-vertex trees with $|\mathcal{H}| \leq (1-o(1))d/2$ and G is any d-regular n-vertex graph with $d \geq (1+o(1))n/2$, then $\mathcal{H}$ packs into G. This yields an approximate version of the tree packing conjecture of Gyárfás and Lehel in the setting of regular host graphs of high degree.

Introduction

Starting with Dirac's theorem on Hamilton cycles, a successful research direction in extremal combinatorics has been to find appropriate minimum degree conditions on a graph G which guarantee the existence of a copy of a (possibly spanning) graph H as a subgraph. On the other hand, several important questions and results in design theory ask for the existence of a decomposition of K_n into edge-disjoint copies of a (possibly spanning) graph H, or more generally into a suitable family of graphs $H_1, \dots, H_t$.

Here, we combine the two directions: rather than finding just a single spanning graph H in a dense graph G, we seek (approximate) decompositions of a dense regular graph G into edge-disjoint copies of spanning sparse graphs H. A specific instance of this is the recent proof of the Hamilton decomposition conjecture and the 1-factorization conjecture for large n [9]. We restrict ourselves to approximate decompositions, but achieve asymptotically best possible results for a much wider class of graphs than matchings and Hamilton cycles.

Previous results: degree conditions for spanning subgraphs

Minimum degree conditions for spanning subgraphs have been obtained mainly for (Hamilton) cycles, trees, factors and bounded degree graphs. Recall that Dirac's theorem states that any n-vertex graph G with minimum degree at least $n/2$ contains a Hamilton cycle. More generally, Abbasi's proof [1] of the El-Zahar conjecture determines the minimum degree threshold for the existence of a copy of H in G where H is a spanning union of vertex-disjoint cycles.

Komlós, Sárközy and Szemerédi [14] proved a conjecture of Bollobás by showing that a minimum degree degree of $n/2 + o(n)$ guarantees every bounded degree n-vertex tree as a subgraph. The Hajnal-Szemerédi theorem implies that the minimum degree threshold for the existence of a K_k-factor is $(1-1/k)n$. The threshold for arbitrary F-factors was determined by Kühn and Osthus [16].

A far-reaching generalisation of the Hajnal-Szemerédi theorem would be provided by the Bollobás-Catlin-Eldridge (BEC) conjecture. This would imply that every n-vertex graph G of minimum degree at least $(1-1/(\Delta+1))n$ contains every n-vertex graph H of maximum degree at most Δ as a subgraph.

Bollobás and Komlós conjectured that one can improve on the BEC-conjecture for graphs H with a linear structure: any n-vertex graph G with minimum degree at least $(1-1/k+o(1))n$ contains a copy of every n-vertex k-chromatic graph H with bounded maximum degree and small bandwidth. Here an n-vertex graph H has *bandwidth* b if there exists an ordering $v_1,\dots,v_n$ of $V(H)$ such that all edges $v_iv_j \in E(H)$ satisfy $|i-j| \le b$. This conjecture was resolved by the bandwidth theorem of Böttcher, Schacht and Taraz [4]. Note that while this result is essentially best possible when considering the class of k-chromatic graphs as a whole (consider e.g. K_k-factors), there are many graphs H for which the actual threshold is significantly smaller (e.g. C_5-factors).

The notion of bandwidth is related to the concept of separability: An n-vertex graph H is said to be *η-separable* if there exists a set S of at most ηn vertices such that every component of $H \setminus S$ has size at most ηn. In general, the notion of having small bandwidth is more restrictive than that of being separable. However, for graphs with bounded maximum degree, it turns out that these notions are actually equivalent (see [3]).

Previous results: (approximate) decompositions into large graphs

As in the previous section, most attention so far has focussed on (Hamilton) cycles, trees, factors, and graphs of bounded degree. Indeed, a classical construction of Walecki going back to the 19th century guarantees a decomposition of K_n into Hamilton cycles whenever n is odd. As mentioned earlier, this was extended to Hamilton decompositions of regular graphs G of high degree by Csaba, Kühn, Lo, Osthus and Treglown [9]. A different generalisation of Walecki's construction is given by the Alspach problem, which asks for a decomposition of K_n into cycles of given length. This was recently resolved by Bryant, Horsley and Petterson [5]. Yet another generalisation would be provided by the Oberwolfach conjecture: given an n-vertex graph H consisting of vertex-disjoint cycles, this asks for a decomposition of K_n into copies of H. This was recently verified for infinitely many n by Bryant and Scharaschkin [6].

A further famous open problem in the area is the tree packing conjecture of Gyárfás and Lehel, which says that for any collection $\mathcal{T} = \{T_1,\dots,T_n\}$ of trees with $|V(T_i)| = i$, the complete graph K_n has a decomposition into $\mathcal{T}$. This was recently proved by Joos, Kim, Kühn and Osthus [13] for the case where n is large and each T_i has bounded degree. The crucial tool was the blow-up lemma for approximate decompositions of ε-regular graphs G by Kim, Kühn, Osthus and Tyomkyn [15]. The special case $G = K_n$ of this lemma implies that if $\mathcal{H}$ is a family of bounded degree n-vertex graphs with $e(\mathcal{H}) \le (1-o(1))\binom{n}{2}$, then K_n has an approximate decomposition into $\mathcal{H}$.

An important type of decomposition of K_n is given by resolvable designs: a resolvable F-design consists of a decomposition into F-factors. Ray-Chaudhuri and Wilson proved the existence of resolvable K_k-designs in K_n (subject to the necessary divisibility conditions being satisfied). This was generalised to arbitrary F-designs by Dukes and Ling [11].

Main result: packing separable graphs of bounded degree

Our main result provides a degree condition which ensures that G has an approximate decomposition into $\mathcal{H}$ for any collection $\mathcal{H}$ of k-chromatic η-separable graphs of bounded degree. In general our degree condition is best possible. By the remark earlier, one can replace the condition of being η-separable by that of having bandwidth at most ηn in Theorem 1. Thus our result implies a version of the bandwidth theorem of [4] in the setting of approximate decompositions.

To state our result, we first introduce the approximate K_k-decomposition threshold δ_k^{reg} for regular graphs.

Definition 1 (Approximate K_k-decomposition threshold for regular graphs)**.** *For each $k \in \mathbb{N}\backslash\{1\}$, let δ_k^{reg} be the infimum over all $\delta \geq 0$ satisfying the following: for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $r \geq \delta n$ every n-vertex r-regular graph G has a K_k-packing consisting of at least $(1-\varepsilon)e(G)/e(K_k)$ copies of K_k.*

Theorem 1 ([8])**.** *For all $\Delta, k \in \mathbb{N}\backslash\{1\}$, $0 < \nu < 1$ and $\max\{1/2, \delta_k^{reg}\} < \delta \leq 1$, there exist $\xi, \eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Suppose that $\mathcal{H}$ is a collection of n-vertex k-chromatic η-separable graphs and G is an n-vertex graph such that*

(i) $(\delta - \xi)n \leq \delta(G) \leq \Delta(G) \leq (\delta + \xi)n$,

(ii) $\Delta(H) \leq \Delta$ *for all* $H \in \mathcal{H}$,

(iii) $e(\mathcal{H}) \leq (1-\nu)e(G)$.

Then $\mathcal{H}$ packs into G.

Note that our result holds for any minor-closed family $\mathcal{H}$ of k-chromatic bounded degree graphs by the separator theorem of Alon, Seymour and Thomas. Moreover, note that since $\mathcal{H}$ may consist e.g. of Hamilton cycles, the condition that G is close to regular is clearly necessary.

Making use of recent bounds for δ_k^{reg} [2, 12, 10, 17], the following corollary is immediate.

Corollary 1 ([8])**.** *For all $\Delta, k \in \mathbb{N}\backslash\{1\}$ and $0 < \nu, \delta < 1$, there exist $\xi > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ the following holds for every n-vertex graph G with*

$$(\delta - \xi)n \leq \delta(G) \leq \Delta(G) \leq (\delta + \xi)n.$$

(i) Let $\mathcal{T}$ be a collection of trees such that for all $T \in \mathcal{T}$ we have $|T| \leq n$ and $\Delta(T) \leq \Delta$. Further suppose $\delta > 1/2$ and $e(\mathcal{T}) \leq (1-\nu)e(G)$. Then $\mathcal{T}$ packs into G.

(ii) Let F be an n-vertex graph consisting of a union of vertex-disjoint cycles and let $\mathcal{F}$ be a collection of copies of F. Further suppose $\delta > 9/10$ and $e(\mathcal{F}) \leq (1-\nu)e(G)$. Then $\mathcal{F}$ packs into G.

(iii) Let $\mathcal{C}$ be a collection of cycles, each on at most n vertices. Further suppose $\delta > 9/10$ and $e(\mathcal{C}) \leq (1-\nu)e(G)$. Then $\mathcal{C}$ packs into G.

(iv) Let n be divisible by k and let $\mathcal{K}$ be a collection of n-vertex K_k-factors. Further suppose $\delta > 1 - 1/(100k)$ and $e(\mathcal{K}) \leq (1-\nu)e(G)$. Then $\mathcal{K}$ packs into G.

Note that (i) can be viewed as an approximate version of the tree packing conjecture in the setting of dense (almost) regular graphs. In a similar sense, (ii) relates to the Oberwolfach conjecture, (iii) relates to the Alspach problem and (iv) relates to the existence of resolvable designs in graphs.

If we drop the assumption of being G close to regular, then one can still ask for the size of the largest packing of bounded degree separable graphs. The following result gives an approximate answer to the above question in the case when $\mathcal{H}$ consists of bipartite graphs.

Theorem 2 ([8])**.** *For all $\Delta \in \mathbb{N}$, $1/2 < \delta \leq 1$ and $\nu > 0$, there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Suppose that $\mathcal{H}$ is a collection of n-vertex η-separable bipartite graphs and G is an n-vertex graph such that*

(i) $\delta(G) \geq \delta n$,

(ii) $\Delta(H) \leq \Delta$ *for all* $H \in \mathcal{H}$,

(iii) $e(\mathcal{H}) \leq \frac{(\delta+\sqrt{2\delta-1}-\nu)n^2}{4}$.

Then $\mathcal{H}$ packs into G.

The result in general cannot be improved: Indeed, for $\delta > 1/2$ the number of edges of the densest regular spanning subgraph of G is close to $(\delta+\sqrt{2\delta-1})n^2/4$ (see [7]). So the bound in (iii) is asymptotically optimal e.g. if n is even and $\mathcal{H}$ consists of Hamilton cycles.

Our main tool will be the recent blow-up lemma for approximate decompositions by Kim, Kühn, Osthus and Tyomkyn [15]: roughly speaking, given a set $\mathcal{H}$ of n-vertex bounded degree graphs and an n-vertex graph G with $e(\mathcal{H}) \leq (1-o(1))e(G)$ consisting of super-regular pairs, it guarantees a packing of $\mathcal{H}$ in G.

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